ON THE SPLITTING METHOD FOR SCHRÖDINGER-LIKE EVOLUTION EQUATIONS

ZDZISLAW BRZEZNIAK AND ANNIE MILLET

Abstract. Using the approach of the splitting method developed by I. Gyöngy and N. Krylov for parabolic quasi linear equations, we study the speed of convergence for general complex-valued stochastic evolution equations. The approximation is given in general Sobolev spaces and the model considered here contains both the parabolic quasi-linear equations under some (non strict) stochastic parabolicity condition as well as linear Schrödinger equations

1. Introduction

Once the well-posedeness of a stochastic differential equation is proved, an important issue is to provide an efficient way to approximate the unique solution. The aim of this paper is to propose a fast converging scheme which gives a simulation of the trajectories of the solution on a discrete time grid, and in terms of some spatial approximation. The first results in this direction were obtained for Stochastic Differential Equations and it is well-known that the limit is sensitive to the approximation. For example, the Stratonovich integral is the limit of Riemann sums with the mid-pint approximation and the Wong Zakai approximation also leads to Stratonovich stochastic integrals, and not to the Itô ones in this finite dimensional framework. There is a huge literature on this topic for stochastic PDEs, mainly extending classical deterministic PDE methods to the stochastic framework. Most of the papers deal with parabolic PDEs and take advantage of the smoothing effect of the second order operator; see e.g. [9], [21], [14], [12], [13], [13], [18] and the references therein. The methods used in these papers are explicit, implicit or Crank-Nicholson time approximations and the space discretization is made in terms of finite differences, finite elements or wavelets. The corresponding speeds of convergence are the "strong" ones, that is uniform in time on some bounded interval $[0, T]$ and with various functional norms for the space variable. Some papers also study numerical schemes in other "hyperbolic" situations, such as the KDV or Schrödinger equations as in [5], [3] and [4]. Let us also mention the weak speed of convergence, that is of an approximation of the expected value of a functional of the solution by the similar one for the scheme obtained by [4] and [6]. These references extend to the

2000 *Mathematics Subject Classification.* Primary 60H15, 65M12; Secondary 65M60, 65M15.
Key words and phrases. Stochastic evolution equations, Schrödinger equation, splitting method, speed of convergence, discretization scheme.

infinite-dimensional setting a very crucial problem for finite-dimensional diffusion processes.

Another popular approach in the deterministic setting, based on semi-groups theory, is the splitting method which solves successively several evolution equations. This technique has been used in a stochastic case in a series of papers by I. Gyöngy and N. Krylov. Let us especially mention reference [10] which uses tools from [20], [15] and [16], and provides a very elegant approach to study quasilinear evolution equations under (non strict) stochastic parabolicity conditions. In their framework, the smoothing effect of the second operator is exactly balanced by the quadratic variation of the stochastic integrals, which implies that there is no increase of space regularity with respect to that of the initial condition. Depending on the number of steps of the splitting, the speed of convergence is at least twice that of the classical finite differences or finite elements methods. A series of papers has been using the splitting technique in the linear and non-linear cases for the deterministic Schrödinger equation; see e.g. [1], [7], [19] and the references therein.

The stochastic Schrödinger equation studies complex-valued processes where the second order operator $i\Delta$ does not improve (nor decay) the space regularity of the solution with respect to that of the initial condition. Well-posedeness of this equation has been proven in a non-linear setting by A. de Bouard and A. Debussche [4]; these authors have also studied finite elements discretization schemes for the corresponding solution under conditions stronger than that in [3].

The aim of this paper is to transpose the approach from [10] to general quasilinear complex-valued equations including both the "classical degenerate" parabolic setting as well as the quasilinear Schödinger equation. Indeed, the method used in [10] consists in replacing the usual splitting via semi-groups arguments by the study of p-th moments of $Z^0 - Z^1$ where Z^0 and Z^1 are solutions of two stochastic evolution equations with the same driving noise and different families of increasing processes V_0^r and V_1^r for $r = 0, 1, \cdots, d_1$ driving the drift term. It does not extend easily to nonlinear drift terms because it is based on some linear interpolation between the two cases V_0^r and V_1^r. Instead of getting an upper estimate of the p-th moment of $Z^0 - Z^1$ in terms of the total variation of the measures defined by the differences $V_0^r - V_1^r$, using integration by parts they obtain an upper estimate in terms of the sup norm of the process $(V_0^r(t) - V_1^r(t), t \in [0, T])$.

We extend this model as follows: given second order linear differential operators L^r, $r = 0, \cdots, d_1$ with complex coefficients, a finite number of sequences of first order linear operators S^l, $l \geq 1$ with complex coefficients, a sequence of real-valued martingales M^l, $l \geq 1$ and a finite number of families of real-valued increasing processes V_i^r, $i \in \{0, 1\}$, $r = 0, \cdots, d_1$, we consider the following system of stochastic evolution equations

$$dZ_i(t) = \sum_r L_r(t,\cdot)Z_i(t)dV_i^r(t) + \sum_l S_l(t,\cdot)Z_i(t)dM^l(t), \quad i = 1, \cdots, d,$$

with an initial condition $Z_i(0)$ belonging to the Sobolev space $H^{m,2}$ for a certain $m \geq 0$. Then under proper assumptions on the various coefficients and processes, under which a stochastic parabolicity condition (see Assumptions **(A1)- (A4(m,p))** in section 2), we prove that for $p \in [2,\infty)$, we have

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|Z_1(t) - Z_0(t)\|_m^p\Big) \leq C\Big(\mathbb{E}\|Z_1(0) - Z_0(0)\|_m^p + A^p\Big), \tag{1.1}$$

where $A = \sup_\omega \sup_{t\in[0,T]} \max_r |V_1^r(t) - V_0^r(t)|$. When the operator $L_r = i\Delta + \tilde{L}^r$ for certain first order differential operator $\tilde{L}_r$, we obtain the quasilinear Schrödinger equation. Note that in this case, the diffusion operators S^l are linear and cannot contain first order derivatives.

As in [10], this abstract result yields the speed of convergence of the following splitting method. Let $\tau_n = \{iT/n, i = 0, \cdots, n\}$ denote a time grid on $[0,T]$ with constant mesh $\delta = T/n$ and define the increasing processes $A_t(n)$ and $B_t(n) = A_{t+\delta}(n)$, where

$$A_t(n) = \begin{cases} k\delta & \text{for } t \in [2k\delta, (2k+1)\delta], \\ t-(k+1)\delta & \text{for } t \in [(2k+1)\delta, (2k+2)\delta]. \end{cases}$$

Given a time-independent second order differential operator L, first order time-independent operators S^l and a sequence $(W^l, l \geq 1)$ of independent one-dimensional Brownian motions, let Z, Z_n and ζ_n be solutions to the evolution equations

$$\begin{aligned} dZ(t) &= LZ(t)dt + \sum_l S_l Z(t) \circ dW_t^l, \\ dZ_n(t) &= LZ_n(t)dA_t(n) + \sum_l S_l Z_n(t) \circ dW_{B_t(n)}^l, \\ d\zeta_n(t) &= L\zeta_n(t)dB_t(n) + \sum_l S_l(t,\cdot)\zeta_n(t) \circ dW_{B_t(n)}^l, \end{aligned}$$

where $\circ dW_t^l$ denotes the Stratonovich integral. The Stratonovich integral is known to be the right one to ensure stochastic parabolicity when the differential operators S_l contain first order partial derivatives (see e.g. [10]). Then $\zeta_n(2k\delta, x) = Z(k\delta, x)$, while the values of $Z_n(2k\delta, x)$ are those of the process $\tilde{Z}_n$ obtained by the following spitting method: one solves successively the correction and prediction equations on each time interval $[iT/n, (i+1)T/n)]$: $dv_t = Lv_t dt$ and then $d\tilde{v}_t = \sum_l S^l(t,\cdot)\tilde{v}_t \circ dW_t^l$. Then, one has $A = CT/n$, and we deduce that $\mathbb{E}\Big(\sup_{t\in[0;T]} \|Z(t) - \tilde{Z}_n(t)\|_m^p\Big) \leq Cn^{-p}$. As in [10], a k-step splitting would yield a rate of convergence of Cn^{-kp}.

The paper is organized as follows. Section 2 states the model, describes the evolution equation, proves well-posedeness as well as apriori estimates. In section 3 we prove (1.1) first in the case of time-independent coefficients of the differential operators, then in the general case under more regularity conditions. As explained above, in section 4 we deduce the rate of convergence of the splitting method for evolution equations generalizing the quasi-linear Schrödinger equation. The speed

of convergence of the non-linear Schrödinger equation will be addressed in a forthcoming paper. As usual, unless specified otherwise, we will denote by C a constant which may change from one line to the next.

2. Well-posedeness and first apriori estimates

2.1. **Well-posedeness results.** Fix $T>0$, $\mathbb{F}=(\mathcal{F}_t, t\in[0,T])$ be a filtration on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ and consider the following $\mathbb{C}$-valued evolution equation on the process $Z(t,x)=X(t,x)+iY(t,x)$ defined for $t\in[0,T]$ and $x\in\mathbb{R}^d$:

$$dZ(t,x)=\sum_{r=0}^{d_1}\big[L_rZ(t,x)+F_r(t,x)\big]\,dV_t^r+\sum_{l\geq 1}\big[S_lZ(t,x)+G_l(t,x)\big]dM_t^l, \tag{2.1}$$

$$Z(0,x)=Z_0(x)=X_0(x)+iY_0(x), \tag{2.2}$$

where d_1 is a positive integer, $(V_t^r, t\in[0,T])$, $r=0,1,\cdots,d_1$ are real-valued increasing processes, $\big(M_t^l, t\in[0,T]\big)$, $l\geq 1$, are independent real-valued $(\mathcal{F}_t, t\in[0,T])$-martingales, L_r (resp. S_l) are second (resp. first) order differential operators defined as follows:

$$\begin{aligned} L_rZ(t,x) &= \sum_{j,k=1}^d D_k\Big(\big[a_r^{j,k}(t,x)+ib_r^{j,k}(t)\big]D_jZ(t,x)\Big)+\sum_{j=1}^d a_r^j(t,x)D_jZ(t,x), \\ &\quad +\big[a_r(t,x)+ib_r(t,x)\big]Z(t,x), \end{aligned} \tag{2.3}$$

$$S_lZ(t,x) = \sum_{j=1}^d \sigma_l^j(t,x)D_jZ(t,x)+\big[\sigma_l(t,x)+i\tau_l(t,x)\big]Z(t,x). \tag{2.4}$$

Let $m\geq 0$ be an integer. Given $\mathbb{C}$-valued functions $Z(.)=X(.)+iY(.)$ and $\zeta(.)=\xi(.)+i\eta(.)$ which belong to $L^2(\mathbb{R}^d)$, let

$$(Z,\zeta)=(Z,\zeta)_0:=\int_{\mathbb{R}^d} Re\big(Z(x)\overline{\zeta(x)}\big)dx=\int_{\mathbb{R}^d}\big[X(x)\xi(x)+Y(x)\eta(x)\big]dx.$$

Thus, we have $(X,\xi)=\int_{\mathbb{R}^d}X(x)\,\xi(x)\,dx$, so that $(Z,\zeta)=(X,\xi)+(Y,\eta)$. For any multi-index $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_d)$ with non-negative integer components α_i, set $|\alpha|=\sum_j\alpha_j$ and for a regular enough function $f:\mathbb{R}^d\to\mathbb{R}$, let $D^\alpha f$ denote the partial derivative $(\frac{\partial}{\partial x_1})^{\alpha_1}\cdots(\frac{\partial}{\partial x_d})^{\alpha_d}(f)$. For $k=1,\cdots,d$, let D_kf denote the partial derivative $\frac{\partial f}{\partial x_k}$. For a $\mathbb{C}$-valued function $F=F_1+iF_2$ defined on $\mathbb{R}^d$, let $D^\alpha F=D^\alpha F_1+iD^\alpha F_2$ and $D_kF=D_kF_1+iD_kF_2$. Finally, given a positive integer m, say that $F\in H^m$ if and only if F_1 and F_2 belong to the (usual) real Sobolev space $H^m=H^{m,2}$. Finally, given $Z=X+iY$ and $\zeta=\xi+i\eta$ which belong to H^m, set

$$(Z,\zeta)_m=\sum_{\alpha:0\leq|\alpha|\leq m}\int_{\mathbb{R}^d}Re\big(D^\alpha Z(x)\overline{D^\alpha\zeta(x)}\big)dx \tag{2.5}$$

$$= \sum_{\alpha:0\le|\alpha|\le m} \int_{\mathbb{R}^d} \big[D^\alpha X(x) D^\alpha \xi(x) + D^\alpha Y(x) D^\alpha \eta(x)\big]\,dx,$$

$$\|Z\|_m^2 = (Z,Z)_m = \sum_{\alpha:|\alpha|\le m} \int_{\mathbb{R}^d} Re\big(D^\alpha Z(x)\overline{D^\alpha Z(x)}\big)\,dx. \tag{2.6}$$

We suppose that the following assumptions are satisfied:

Assumption (A1) For $r = 0,\cdots,d_1$, $(V_t^r, t\in[0,T])$ are predictable increasing processes. There exist a positive constant $\tilde{K}$ and an increasing predictable process $(V_t, t\in[0,T])$ such that:

$$V_0 = V_0^r = 0,\ r=0,\cdots d_1, \quad V_T \le \tilde{K} \text{ a.s.}, \tag{2.7}$$

$$\sum_{r=0}^{d_1} dV_t^r + \sum_{l\ge1} d\langle M^l\rangle_t \le dV_t \quad \text{a.s. in the sense of measures.} \tag{2.8}$$

Assumption (A2)

(i) For $r = 0,1,\cdots d_1$, the matrices $(a_r^{j,k}(t,x), j,k=1,\cdots d)$ and $(b_r^{j,k}(t), j,k = 1,\cdots d)$ are $(\mathcal{F}_t)$-predictable real-valued symmetric for almost every ω, $t\in[0,T]$ and $x\in\mathbb{R}^d$.

(ii) For every $t\in(0,T]$ and $x,y\in\mathbb{R}^d$

$$\sum_{j,k=1}^{d} y^j y^k \Big[2\sum_{r=0}^{d_1} a_r^{j,k}(t,x)\,dV_t^r - \sum_{l\ge1} \sigma_l^j(t,x)\sigma_l^k(t,x)d\langle M^l\rangle_t\Big] \ge 0 \tag{2.9}$$

a.s. in the sense of measures.

Assumption (A3(m)) There exists a constant $\tilde{K}(m)$ such that for all $j,k=1,\cdots d$, $r=0,\cdots,d_1$, $l\ge1$, any multi-indices α (resp. β) of length $|\alpha|\le m+1$ (resp. $|\beta|\le m$), and for every $(t,x)\in(0,T]\times\mathbb{R}^d$ one has a.s.

$$|D^\alpha a_r^{j,k}(t,x)| + |b^{j,k}(t)| + |D^\alpha a_r^j(t,x)| + |D^\beta a_r(t,x)| + |D^\beta b_r(t,x)| \le \tilde{K}(m), \tag{2.10}$$

$$|D^\alpha \sigma_l^j(t,x)| + |D^\alpha \sigma_j(t,x)| + |D^\alpha \tau_l(t,x)| \le \tilde{K}(m). \tag{2.11}$$

Assumption (A4(m,p)) Let $p\in[2,+\infty)$; for any $r=0,\cdots,d_1$, $l\ge1$, the processes $F_r(t,x) = F_{r,1}(t,x) + iF_{r,2}(t,x)$ and $G_l(t,x) = G_{l,1}(t,x) + iG_{l,2}(t,x)$ are predictable, $F_r(t,\cdot)\in H^m$ and $G_r(t,\cdot)\in H^{m+1}$. Furthermore, if we denote

$$K_m(t) = \int_0^t \Big[\sum_{r=0}^{d_1} \|F_r(s)\|_m^2\,dV_s^r + \sum_{l\ge1} \|G_l(s)\|_{m+1}^2 d\langle M^l\rangle_s\Big], \tag{2.12}$$

then

$$\mathbb{E}\big(\|Z_0\|_m^p + K_m^{\frac{p}{2}}(T)\big) < +\infty. \tag{2.13}$$

The following defines what is considered to be a (probabilistically strong) weak solution of the evolution equations (2.1)-(2.2).

Definition 1. *A $\mathbb{C}$-valued $(\mathcal{F}_t)$-predictable process Z is a solution to the evolution equation* (2.1) *with initial condition Z_0 if*

$$\mathbb{P}\Big(\int_0^T \|Z(s)\|_1^2 ds < +\infty\Big) = 1, \quad \mathbb{E}\int_0^T |Z(s)|^2 dV_s < \infty,$$

and for every $t \in [0,T]$ and $\Phi = \phi + i\psi$, where ϕ and ψ are $\mathcal{C}^\infty$ functions with compact support from $\mathbb{R}^d$ to $\mathbb{R}$, one has a.s.

$$\begin{aligned}(Z(t),\Phi) = (Z(0),\Phi) &+ \sum_{r=0}^{d_1}\int_0^t \Big[-\sum_{j,k=1}^d \Big(\big[a_r^{j,k}(s,\cdot) + i b_r^{j,k}(s)\big] D_j Z(s,\cdot), D_k\Phi\Big) \\ &+ \sum_{j=1}^d \Big(a_r^j(s,\cdot) D_j Z(s,\cdot) + [a_r(s,\cdot) + i b_r(s,\cdot)] Z(s,\cdot) + F_r(s,\cdot), \Phi\Big)\Big] dV_r(s) \\ &+ \sum_{l\geq 1}\int_0^t \big(S_l(Z(s,\cdot)) + G_l(s,\cdot), \Phi\big)\, dM_s^l. \end{aligned} \tag{2.14}$$

Theorem 1. *Let $m \geq 1$ be an integer and suppose that Assumptions* **(A1),(A2), (A3(m))** *and* **(A4(m,2))** *(i.e., for $p = 2$) are satisfied.*

(i) Then equations (2.1) *and* (2.2) *have a unique solution Z, such that*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|Z(t,\cdot)\|_m^2\Big) \leq C\,\mathbb{E}\Big(\|Z_0\|_m^2 + K_m(T)\Big) < \infty, \tag{2.15}$$

for a constant C that only depends on the constants which appear in the above listed conditions. Almost surely, $Z \in \mathcal{C}([0,T], H^{m-1})$ and almost surely the map $[0,T] \ni t \mapsto Z(t,\cdot) \in H^m$ is weakly continuous.

(ii) Suppose furthermore that Assumption **(A4(m,p))** *holds for $p \in (2,+\infty)$. Then there exists a constant C_p as above such that*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|Z(t,\cdot)\|_m^p\Big) \leq C_p\,\mathbb{E}\Big(\|Z_0\|_m^p + K_m^{p/2}(T)\Big). \tag{2.16}$$

Proof. Set $\mathcal{H} = H^m$, $\mathcal{V} = H^{m+1}$ and $\mathcal{V}' = H^{m-1}$. Then $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ is a Gelfand triple for the equivalent norm $|(I-\Delta)^{m/2}u|_{L^2}$ on the space H_m. Given $Z = X+iY \in \mathcal{V}$ and $\zeta = \xi + i\eta \in \mathcal{V}'$ set

$$\langle Z,\zeta\rangle_m = \langle X,\xi\rangle_m + \langle Y,\eta\rangle_m, \quad \text{and } \langle Z,\zeta\rangle := \langle Z,\zeta\rangle_0,$$

where $\langle X,\xi\rangle_m$ and $\langle Y,\eta\rangle_m$ denote the duality between the (real) spaces H^{m+1} and H^{m-1}. For every multi-index α, let

$$\mathcal{I}(\alpha) = \{(\beta,\gamma)\ :\ \alpha = \beta+\gamma,\ |\beta|,|\gamma| \in \{0,\cdots,|\alpha|\}\}.$$

To ease notations, we skip the time parameter when writing the coefficients a_r, b_r, σ_l and τ_l. Then for $l \geq 1$, using the Assumption **(A3(m))**, we deduce that if

$Z=X+iY\in H^{m+1}$, we have for $|\alpha|\le m$,

$$D^\alpha[S_l(Z)]=\sum_{j=1}^d \sigma_l^j(x)\,D_jD^\alpha Z+\sum_{(\beta,\gamma)\in\mathcal{I}(\alpha)} C_l(\beta,\gamma)D^\beta Z, \tag{2.17}$$

with functions $C_l(\beta,\gamma)$ from $\mathbb{R}^d$ to $\mathbb{C}$ such that $\sup_{l\ge 1}\sup_{x\in\mathbb{R}^d}|C_l(\beta,\gamma)(x)|<+\infty$. A similar computation proves that for every multi-index α with $|\alpha|\le m$, $r=0,\cdots,d_1$

$$D^\alpha[L_r(Z)]=L_r(D^\alpha Z)+\sum_{j,k=1}^d\sum_{(\beta,\gamma)\in\mathcal{I}(\alpha):|\gamma|=1} D_k\Big(D^\gamma a_r^{j,k}D_jD^\beta Z\Big)+\sum_{|\beta|\le m}C_r(\beta,\gamma)D^\beta Z, \tag{2.18}$$

for some bounded functions $C_r(\beta,\gamma)$ from $\mathbb{R}^d$ into $\mathbb{C}$. Hence for every $r=0,\cdots,d_1$, one has a.s. $L_r:\mathcal{V}\times\Omega\to\mathcal{V}'$ and similarly, for every $l\ge 1$, a.s. $S_l:\mathcal{V}\times\Omega\to\mathcal{H}$.

For every $\lambda>0$ and $Z=X+iY\in\mathcal{H}$, let us set

$$L_{r,\lambda}Z:=L_rZ+\lambda(\Delta X+i\Delta Y)=L_rZ+\lambda\Delta Z. \tag{2.19}$$

Consider the evolution equation for the process $Z^\lambda(t,x)=X^\lambda(t,x)+iY^\lambda(t,x)$,

$$dZ^\lambda(t,x)=\sum_{r=0}^{d_1}\big[L_{r,\lambda}Z^\lambda(t,\cdot)+F_r(t,x)\big]\,dV_t^r+\sum_{l\ge1}\big[S_lZ^\lambda(t,x)+G_l(t,x)\big]dM_t^l, \tag{2.20}$$

$$Z^\lambda(0,x)=Z_0(x)=X_0(x)+iY_0(x). \tag{2.21}$$

In order to prove well-posedeness of the problem (2.20)-(2.21), firstly we have to check the following stochastic parabolicity condition:

Condition (C1) There exists a constant $K>0$ such that for $Z\in H^{m+1}$, $t\in[0,T]$:

$$2\sum_{r=0}^{d_1}\langle L_rZ\,,\,Z\rangle_m\,dV_t^r+\sum_{l\ge0}\|S_l(Z)\|_m^2\,d\langle M^l\rangle_t\le K\|Z\|_m^2\,dV_t$$

a.s. in the sense of measures.

Let $Z=X+iY\in H^{m+1}$; using (2.18) and (2.17), we deduce that

$$2\sum_{|\alpha|=m}\sum_{r=0}^{d_1}\langle D^\alpha L^rZ\,,\,D^\alpha Z\rangle\,dV_t^r+\sum_{|\alpha|=m}\sum_{l\ge1}|D^\alpha S_lZ|^2\,d\langle M^l\rangle_t=\sum_{\kappa=1}^5 dT_\kappa(t), \tag{2.22}$$

where to ease notation we drop the time index in the right handsides and we set:

$$\begin{aligned}dT_1(t)=\sum_{|\alpha|=m}\sum_{j,k=1}^d\Big\{&-2\sum_{r=1}^{d_1}\Big[\Big(a_r^{j,k}D_jD^\alpha X,D_kD^\alpha X\Big)-\Big(b_r^{j,k}D_jD^\alpha Y,D_kD^\alpha X\Big)\\&+\Big(a_r^{j,k}D_jD^\alpha Y,D_kD^\alpha Y\Big)+\Big(b_r^{j,k}D_jD^\alpha X,D_kD^\alpha Y\Big)\Big]dV_t^r\\&+\sum_{l\ge1}\Big[\Big(\sigma_l^jD_jD^\alpha X,\sigma_l^kD_kD^\alpha X\Big)+\Big(\sigma_l^jD_jD^\alpha Y,\sigma_l^kD_kD^\alpha Y\Big)\Big]d\langle M^l\rangle_t\Big\},\end{aligned}$$

$$
\begin{aligned}
dT_2(t) = &- 2\sum_{r=0}^{d_1} \sum_{|\alpha|\le m} \Big\{ \sum_{j,k=1}^{d} \sum_{(\beta,\gamma)\in\mathcal{I}(\alpha):|\gamma|=1} \\
&\qquad \Big[\big(D^\gamma a_r^{j,k} D_j D^\beta X, D_k D^\alpha X\big) + \big(D^\gamma a_r^{j,k} D_j D^\beta Y, D_k D^\alpha Y\big)\Big] \\
&+ \sum_{j=1}^{d} \Big[\big(a_r^j D_j D^\alpha X, D^\alpha X\big) + \big(a_r^j D_j D^\alpha Y, D^\alpha Y\big)\Big]\Big\} dV_t^r, \\
dT_3(t) = &\sum_{l\ge 1} \sum_{j,k=1}^{d} \sum_{|\alpha|=m} \sum_{(\beta,\gamma)\in\mathcal{I}(\alpha):|\gamma|=1} \Big[\big(D^\gamma \sigma_l^j D_j D^\beta X, \sigma_l^k D_k D^\alpha X\big) \\
&+ \big(D^\gamma \sigma_l^j D_j D^\beta Y, \sigma_l^k D_k D^\alpha Y\big) d\langle M^l\rangle_t, \\
dT_4(t) = &\sum_{l\ge 1} \sum_{j,k=1}^{d} \sum_{|\alpha|=m} \Big[(\sigma_l^j D_j D^\alpha X, \sigma_l D^\alpha X) - (\sigma_l^j D_j D^\alpha X, \tau_l D^\alpha Y) \\
&+ (\sigma_l^j D_j D^\alpha Y, \tau_l D^\alpha X) + (\sigma_l^j D_j D^\alpha Y, \sigma_l D^\alpha Y)\Big] d\langle M^l\rangle_t, \\
dT_5(t) = &\sum_{|\alpha|\vee|\beta|\le m} \Big\{ \sum_{r=0}^{d_1} \Big[\sum_{j,k=1}^{d} \big(C_r^{j,k}(.) D^\beta Z, D^\alpha Z\big) \\
&+ \sum_{j=1}^{d} \big(C_r^j(.) D^\beta Z, D^\alpha Z\big) + \big(C_r(.) D^\beta Z, D^\alpha Z\big)\Big] dV_t^r \\
&+ \sum_{l\ge 1}\Big[\sum_{j=1}^{d} \Big(\Big[\tilde{C}_l(.) + \sum_{l\ge 1} \tilde{C}_l^j(.)\Big]] D^\beta Z, D^\alpha Z\Big)\Big] d\langle M^l\rangle_t \Big\},
\end{aligned}
$$

where $C_r^{j,k}$, C_r^j, C_r, $\tilde{C}_l^j$ and $\tilde{C}_l$ are bounded functions from $\mathbb{R}^d$ to $\mathbb{C}$ due to Assumption **(A4(m,p))** for any $p\in[2,\infty)$.

For every r the matrix b_r is symmetric; hence $\sum_{j,k}\big[(b_r^{j,k} D_j D^\alpha X, D_k D^\alpha Y) - (b_r^{j,k} D_j D^\alpha Y, D_k D^\alpha X)\big] = 0$. Hence, Assumption **(A2)** used with $y_j = D_j D^\alpha X$ and with $y_j = D_j D^\alpha Y$, $j = 1,\cdots,d$, implies $T_1(t)\le 0$ for $t\in[0,T]$. Furthermore, Assumption **(A3(m))** yields the existence of a constant $C>0$ such that $dT_5(t)\le C\|Z(t)\|_m^2 dV_t$ for all $t\in[0,T]$. Integration by parts shows that for regular enough functions $f,g,h:\mathbb{R}^d\to\mathbb{R}$, $(\beta,\gamma)\in\mathcal{I}(\alpha)$ with $|\gamma|=1$, we have

$$
(f D^\beta g, D^\alpha h) = -(f D^\alpha g, D^\beta h\rangle - \langle D^\gamma f D^\beta g, D^\beta h\rangle. \tag{2.23}
$$

Therefore, the symmetry of the matrices a_r implies that for $\phi\in\{X(t),Y(t)\}$ and $r=0,\cdots,d_1$,

$$
\sum_{j,k=1}^{d} \big(D^\gamma a_r^{j,k} D_j D^\beta \phi, D_k D^\alpha \phi\big) = -\frac{1}{2}\sum_{j,k=1}^{d} \big(D^\gamma (D^\gamma a_r^{j,k}) D_j D^\beta \phi, D_k D^\beta \phi\big).
$$

A similar argument proves that for fixed $j=1,\cdots,d$, $r=0,\cdots,d_1$ and $\phi=X(t)$ or $\phi=Y(t)$,

$$(a_r^j D_j D^\alpha\phi, D^\alpha\phi) = -\frac{1}{2}(D_j a_r^j D^\alpha\phi, D^\alpha\phi).$$

Therefore, Assumption **(A3(0))** implies the existence of $K>0$ such that $dT_2(t)\leq K\|Z(t)\|_m^2 dV_t$ for all $t\in[0,T]$. Furthermore, $dT_4(t)$ is the sum of terms $(\phi\psi, D_j\phi)\, d\langle M^l\rangle_t$ where $\phi = D^\alpha X(t)$ or $\phi = D^\alpha Y(t)$, and $\psi = fg$, with $f\in\{\sigma_l^k\}$ and $g\in\{\sigma_l,\tau_l\}$. The identity $(\phi\psi, D_j\phi) = -\frac{1}{2}(\phi D_j\psi,\phi)$, which is easily deduced from integration by parts, and Assumptions **(A1)** and **(A3(m))** imply the existence of $K>0$ such that $dT_4(t)\leq K\|Z(t)\|_m^2 dV_t$ for every $t\in[0,T]$. The term $dT_3(t)$ is the sum over $l\geq 1$ and multi-indices α with $|\alpha|=m$ of

$$A(l,\alpha) = \sum_{j,k=1}^d \sum_{(\beta,\gamma)\in\mathcal{I}(\alpha):|\gamma|=1} \big(D^\gamma f_l^j f_l^k D_j D^\beta\varphi, D_k D^\alpha\varphi\big),$$

with $\varphi = X(t)$ or $\varphi = Y(t)$ and $f_l^j = \sigma_l^j$ for every $j=1,\cdots,d$. Then, $A(l,\alpha) = B(l,\alpha) - C(l,\alpha)$, where $B(l,\alpha) = \sum_{j,k=1}^d B_{j,k}(l,\alpha)$ and

$$B_{j,k}(l,\alpha) = \sum_{(\beta,\gamma)\in\mathcal{I}(\alpha):|\gamma|=1} \big(D^\gamma(f_l^j f_l^k) D_j D^\beta\varphi, D_k D^\alpha\varphi\big),$$

$$C(l,\alpha) = \sum_{j,k=1}^d \sum_{(\beta,\gamma)\in\mathcal{I}(\alpha):|\gamma|=1} \big(D^\gamma f_l^k f_l^j D_j D^\beta\varphi, D_k D^\alpha\varphi\big).$$

Integrating by parts twice and exchanging the partial derivatives D_j and D_k in each term of the sum in $C(l,\alpha)$, we deduce that

$$\begin{aligned}
\big(D^\gamma f_l^k f_l^j D_j D^\beta\varphi, D_k D^\alpha\varphi\big) &= -\big(\big[D_k[D^\gamma f_l^k f_l^j] D_j D^\beta\varphi + D^\gamma f_l^k f_l^j D_k D_j D^\beta\varphi\big], D^\alpha\varphi\big)\\
&= \big(-D_k[D^\gamma f_l^k f_l^j] D_j D^\beta\varphi + \big(D_j[D^\gamma f_l^k f_l^j] D_k D^\beta\varphi, D^\alpha\varphi\big)\\
&\quad + \big(D^\gamma f_l^k f_l^j D_k D^\beta\varphi, D_j D^\alpha\varphi\big).
\end{aligned}$$

On the other hand, by symmetry we obviously have

$$\sum_{j,k} \big(D^\gamma f_l^j f_l^k D_k D^\beta\varphi, D_j D^\alpha\varphi\big) = \sum_{j,k} \big(D^\gamma f_l^k f_l^j D_j D^\beta\varphi, D_k D^\alpha\varphi\big).$$

Using Assumptions **(A1)** and **(A3(1))** we deduce that there exist bounded functions $\phi(\alpha,\tilde\alpha,l)$ defined for multi-indices $\tilde\alpha$ which have at most one component different from those of α, and such that

$$A(l,\alpha) = \frac{1}{2}B(l,\alpha) + \sum_{|\tilde\alpha|=m} \Big(\phi(\alpha,\tilde\alpha,l) D^{\tilde\alpha}\Phi, D^\alpha\Phi\Big).$$

Furthermore, integration by parts yields

$$\sum_{j,k=1}^{d} B_{j,k}(l,\alpha) = -\frac{1}{2}\sum_{j,k=1}^{d}\sum_{(\beta,\gamma)\in\mathcal{I}(\alpha):|\gamma|=1}\big(D^\gamma D^\gamma[f_l^j f_l^k]D_j D^\beta\varphi, D_k D^\beta\varphi\big).$$

Thus, Assumption **(A3(1))** implies the existence of a constant $C>0$ such that for the various choices of φ and f_l^k, $\sum_{l\geq 1}\sum_{|\alpha|=m}|B(l,\alpha)|d\langle M^l\rangle_t \leq C\|Z(t)\|_m^2 dV_t$ for every $t\in[0,T]$. Therefore, we deduce that we can find a constant $K>0$ such that $dT_3(t)\leq K\|Z(t)\|_m^2 dV_t$. The above inequalities and (2.22) complete the proof of Condition **(C1)**.

Since L_r are linear operators, Condition **(C1)** implies the following classical Monotonicity, Coercivity and Hemicontinuity: for every $Z,\zeta\in H^{m+1}$ and $L_{r,\lambda}$ defined by (2.19),

$$2\sum_{r=0}^{d_1}\langle L_r Z - L_r\zeta\,,\, Z-\zeta\rangle_m dV_t^r + \sum_{l\geq 1}\|S_l(Z)-S_l(\zeta)\|_m^2 d\langle M^l\rangle_t \leq K\|Z-\zeta\|_m^2 dV_t,$$

$$2\sum_{r=0}^{d_1}\langle L_{r,\lambda}Z\,,\, Z\rangle_m dV_t^r + \sum_{l\geq 1}\|S_l(Z)\|_m^2 d\langle M^l\rangle_t + 2\lambda\|Z\|_{m+1}^2\sum_{r=0}^{d_1} dV_t^r \leq K\|Z\|_m^2 dV_t$$

a.s. in the sense of measures, and for $Z_i\in H^{m+1}$, $i=1,2,3$, $r=0,\cdots,d_1$ and $\lambda>0$, the map $a\in\mathbb{C}\to\langle L_{r,\lambda}(Z_1+aZ_2\,,\,Z_3\rangle_m$ is continuous.

The following condition **(C2)** gathers some useful bounds on the operators L_r and S_l for $0\leq r\leq d_1$ and $l\geq 1$.

Condition (C2) There exist positive constants $K_i, i=2,3,4$ such that for $Z\in H^{m+1}$, $\lambda\in[0,1]$, $r=0,\cdots,d_1$ and $l\geq 1$:

$$2\|L_{r,\lambda}Z\|_{m-1}^2 + \|S_l Z\|_m^2 \leq K_2\|Z\|_{m+1}^2 \text{ a.s.} \tag{2.24}$$

$$|(S_l Z, Z)_m| \leq K_3\|Z\|_m^2 \text{ and } |(S_l Z, G_l)_m| \leq K_4\|Z\|_m\|G_l\|_{m+1} \text{ a.s..} \tag{2.25}$$

The inequality (2.24) is a straightforward consequence of the Cauchy-Schwarz inequality and of Assumption **(A3(m))**. Using integration by parts and Assumptions **(A3(m))-(A4(m,p))**, we deduce that if $G_l(t)=G_{l,1}(t)+iG_{l,2}(t)$,

$$\begin{aligned}|(S_l Z, Z)_m| \leq & \frac{1}{2}\sum_{|\alpha|=m}\sum_{j=1}^{d}\big|\big[\big(D_j\sigma_l^j D^\alpha X, D^\alpha X\big)+\big(D_j\sigma_l^j D^\alpha Y, D^\alpha Y\big)\big]\big|\\ &+\sum_{|\alpha|\vee|\beta|\leq m}\big|\big(C(\alpha,\beta,l)D^\alpha Z, D^\beta Z\big)\big|\end{aligned}$$

$$|(S_l Z, G_l)_m| \leq \sum_{j=1}^{d}\sum_{|\alpha|=m}\big[\big|\big(\sigma_l^j D^\alpha X, D_j D^\alpha G_{l,1}\big)\big| + \big|\big(\sigma_l^j D^\alpha Y, D_j D^\alpha G_{l,2}\big)\big|$$

$$+\sum_{|\alpha|\vee|\beta|\le m}\big|\big(C(\alpha,\beta,l)D^\alpha Z, D^\beta G_l)\big)\big|,$$

for constants $C(\alpha,\beta,l)$, α,β,l such that $\sup_{\alpha,\beta,l} C(\alpha,\beta,l)\le C<\infty$. Hence a simple application of the Cauchy-Schwarz and Young inequalities implies inequality (2.25).

We then proceed as in the proof of Theorem 3.1 in [16] for fixed $\lambda>0$ (see also [20] and [10]). To ease notations, we do not write the Galerkin approximation as the following estimates would be valid with constants which do not depend on the dimension of the Galerkin approximation, and hence would still be true for the weak and weak* limit in $L^2([0,T]\times\Omega;H_{m+1})$ and $L^2(\Omega;L^\infty(0,T;H_m))$. Let us fix a real number $N>0$ and let $\tau_N=\inf\{t\ge 0: \|Z^\lambda(t)\|_m\ge N\}\wedge T$. The Itô formula, the stochastic parabolicity condition **(C1)** and the Davies inequality imply that for any $t\in[0,T]$ and $\lambda\in(0,1]$,

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{s\in[0,t]}\|Z^\lambda(s\wedge\tau_N)\|_m^2\Big)+2\lambda\mathbb{E}\int_0^{t\wedge\tau_N}\|Z^\lambda(s)\|_{m+1}^2ds\le\mathbb{E}\|Z_0\|_m^2\\
&\quad+2\sum_{r=0}^{d_1}\mathbb{E}\int_0^t\big|\langle F_r(s\wedge\tau_N),Z^\lambda(s\wedge\tau_N)\rangle_m\big|dV_s^r\\
&\quad+\sum_{l\ge1}\mathbb{E}\int_0^t\big[2\big|\big(S_lZ^\lambda(s\wedge\tau_N),G_l(s\wedge\tau_N)\big)_m\big|+\|G_l(s\wedge\tau_N)\|_m^2\big]d\langle M^l\rangle_s\\
&\quad+6\,\mathbb{E}\Big(\sum_{l\ge1}\Big\{\int_0^t\big(S_lZ^\lambda(s\wedge\tau_N)+G_l(s\wedge\tau_N),Z^\lambda(s\wedge\tau_N)\big)_m^2d\langle M^l\rangle_s\Big\}^{\frac12}\Big)
\end{aligned}$$

The Cauchy-Schwarz inequality, the upper estimate (2.25) in Condition **(C2)** and inequalities (2.7) - (2.8) in Assumption **(A1)** imply the existence of some constant $K>0$ such that for any $\delta>0$,

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{s\in[0,t]}\|Z^\lambda(s\wedge\tau_N)\|_m^2\Big)+2\lambda\mathbb{E}\int_0^{t\wedge\tau_N}\|Z^\lambda(s)\|_{m+1}^2ds\le\mathbb{E}\|Z_0\|_m^2\\
&\quad+3\,\delta\mathbb{E}\Big(\sup_{s\in[0,t]}\|Z^\lambda(s\wedge\tau_N)\|_m^2\Big)+\tilde K\delta^{-1}\sum_{r=0}^{d_1}\mathbb{E}\int_0^t\|F_r(s)\|_m^2dV_s^r\\
&\quad+\mathbb{E}\int_0^t\sum_{l\ge1}K\big[\delta^{-1}+1\big]\|G_l(s)\|_{m+1}^2d\langle M^l\rangle_s+\mathbb{E}\int_0^t\|Z^\lambda(s\wedge\tau_N)\|_m^2dV_s.
\end{aligned}$$

For $\delta=\frac16$, the Gronwall Lemma implies that for some constant C we have for all $N>0$ and $\lambda\in(0,1]$,

$$\mathbb{E}\Big(\sup_{s\in[0,t]}\|Z^\lambda(s\wedge\tau_N)\|_m^2\Big)+\lambda\mathbb{E}\int_0^{t\wedge\tau_N}\|Z^\lambda(s)\|_{m+1}^2ds\le C\mathbb{E}\big(\|Z_0\|_m^2+K_m(T)\big).$$

As $N \to \infty$, we deduce that $\tau_N \to \infty$ a.s. and by the monotone convergence theorem,

$$\mathbb{E}\Big(\sup_{s\in[0,T]} \|Z^\lambda(s)\|_m^2\Big) + \lambda \mathbb{E}\int_0^T \|Z^\lambda(s)\|_{m+1}^2 ds \le C\mathbb{E}\big(\|Z_0\|_m^2 + K_m(T)\big).$$

Furthermore, Z^λ belongs a.s. to $\mathcal{C}([0,T],H^m)$. As in [16], we deduce the existence of a sequence $\lambda_n \to 0$ such that $Z^{\lambda_n} \to Z$ weakly in $L^2([0,T]\times\Omega;H^m)$. Furthermore, Z is a solution to (2.1) and (2.2) such that (2.15) holds and is a.s. weakly continuous from $[0,T]$ to H^m.

The uniqueness of the solution follows from the growth condition (2.24) in **(C2)** and the monotonicity condition which is deduced from the stochastic parabolicity property **(C1)**.

(ii) Suppose that Assumption **(A4(m,p))** holds for $p\in[2,\infty)$. Set $p=2\tilde{p}$ with $\tilde{p}\in[1,\infty)$; the Itô formula, the stochastic parabolicity condition **(C1)**, the growth conditions **(C2)**, the Burkholder-Davies-Gundy and Schwarz inequalities yield the existence of some constant C_p which also depends in the various constants in Assumptions **(A1)-(A4(m,p))**, and conditions **(C1)-(C2)**, such that:

$$\begin{aligned}\mathbb{E}\Big(\sup_{s\in[0,t]}\|Z(s)\|_m^p\Big) &\le C_p\Big[\mathbb{E}\|Z_0\|_m^p + \mathbb{E}\Big(\sup_{s\in[0,t]}\|Z(s)\|_m^{\tilde{p}}\Big|\sum_{r=0}^{d_1}\int_0^t \|F_r(s)\|_m dV_s^r\Big|^{\tilde{p}}\Big)\\
&+\mathbb{E}\Big(\sup_{s\in[0,t]}\|Z(s)\|_m^{\tilde{p}}\Big|\sum_{l\ge1}\int_0^t \|G_l(s)\|_{m+1} d\langle M^l\rangle_s\Big|^{\tilde{p}}\Big)\\
&+\mathbb{E}\Big(\sup_{s\in[0,t]}\|Z(s)\|_m^{\tilde{p}}\Big|\sum_{l\ge1}\int_0^t \big[\|Z(s)\|_m^2+\|G_l(s)\|_m^2\big] d\langle M^l\rangle_s\Big|^{\tilde{p}/2}\Big).\end{aligned}$$

Using the Hölder and Young inequalities, (2.13) as well as Assumptions **(A1)** we deduce the existence of a constant $K>0$ such that for any $\delta>0$

$$\begin{aligned}\mathbb{E}\Big(\sup_{s\in[0,t]}\|Z(s)\|_m^p\Big) &\le 3\delta\mathbb{E}\Big(\sup_{s\in[0,t]}\|Z(s)\|_m^p\Big) + C(p)\Big[\mathbb{E}\|Z_0\|_m^p\\
&+K^{\tilde{p}}\delta^{-1}\mathbb{E}\Big|\int_0^t\sum_{r=0}^{d_1}\|F_r(s)\|_m^2 dV_s^r\Big|^{\tilde{p}} + K^{\tilde{p}}\delta^{-1}\mathbb{E}\Big|\int_0^t\sum_{l\ge1}\|G_l(s)\|_{m+1}^2 d\langle M^l\rangle_s\Big|^{\tilde{p}}\\
&+K^{\tilde{p}-1}\delta^{-1}\mathbb{E}\int_0^t\|Z(s)\|_m^p dV_s + \delta^{-1}\Big|\mathbb{E}\int_0^t\sum_{l\ge1}\|G_l(s)\|_m^2 d\langle M^l\rangle_s\Big|^{\tilde{p}}\Big].\end{aligned}$$

Let $\delta=\frac{1}{6}$ and introduce the stopping time $\tau_N=\inf\{t\ge0:\|Z(t)\|_m\ge N\}\wedge T$. Replacing t by $t\wedge\tau_N$ in the above upper estimates, the Gronwall Lemma and (2.13) prove that there exists a constant C such that $\mathbb{E}\Big(\sup_{s\in[0,t]\wedge\tau_N}\|Z(s)\|_m^{2p}\Big)\le C\mathbb{E}\big(\|Z_0\|_m^p+K_m(T)^{p/2}\big)$ for every $N>0$. As $N\to\infty$ the monotone convergence theorem concludes the proof of (2.16). This ends of the proof of Theorem 1. $\square$

Remark 2.1. *If $a_r^{j,k}(t,x)=0$, for example for the Schrödinger equation, Assumption* **(A2)** *implies that $\sigma_l^j=0$.*

2.2. **Further a priori estimates on the difference.** Theorem 1 is used to upper estimate moments of the difference of two processes solutions to equations of type (2.1). For $\varepsilon=0,1$, $r=0,\cdots,d_1$, $l\geq 1$, $j,k=1,\cdots,d$, let $a_{\varepsilon,r}^{j,k}(t,x)$, $b_{\varepsilon,r}^{j,k}(t)$, $a_{\varepsilon,r}^{j}(t,x)$, $a_{\varepsilon,r}(t,x)$, $b_{\varepsilon,r}(t,x)$, $\sigma_{\varepsilon,l}^{j}(t,x)$, $\sigma_{\varepsilon,l}(t,x)$, $\tau_{\varepsilon,l}(t,x)$ be coefficients, $F_{\varepsilon,r}(t,x)$, $G_{\varepsilon,l}(t,x)$ be processes, and let $Z_{\varepsilon,0}$ be random variables which satisfy the assumptions **(A1)**-**(A3(m))** and **(A4(m,p))** for some $m\geq 1$, $p\in[2,\infty)$, the same martingales $(M_t^l, t\in[0,T])$ and increasing processes $(V_t^r, t\in[0,T])$. Let $L_{\varepsilon,r}$ and $S_{\varepsilon,l}$ be defined as in (2.3) and (2.4) respectively. Extend these above coefficients, operators, processes and random variables to $\varepsilon\in[0,1]$ as follows: if f_0 and f_1 are given, for $\varepsilon\in[0,1]$, let $f_\varepsilon=\varepsilon f_1+(1-\varepsilon)f_0$. Note that by convexity, all the previous assumptions are satisfied for any $\varepsilon\in[0,1]$. Given $\varepsilon\in[0,1]$, let Z_ε denote the solution to the evolution equation: $Z_\varepsilon(0,x)=Z_{\varepsilon,0}(x)$ and

$$dZ_\varepsilon(t,x)=\sum_{r=0}^{d_1}\big[L_{\varepsilon,r}Z_\varepsilon(t,x)+F_{\varepsilon,r}(t,x)\big]\,dV_t^r+\sum_{l\geq 1}\big[S_{\varepsilon,l}Z_\varepsilon(t,x)+G_{\varepsilon,l}(t,x)\big]\,dM_t^l. \tag{2.26}$$

Thus, Theorem 1 immediatly yields the following

Corollary 1. *With the notations above, the solution Z_ε to* (2.26) *with the initial condition $Z_{\varepsilon,0}$ exists and is unique with trajectories in $C([0,T];H^{m-1})\cap L^\infty(0,T;H^m)$. Furthermore, the trajectories of Z_ε belong a.s. to $C_w([0,T];H^m)$ and there exists a constant $C_p>0$ such that*

$$\sup_{\varepsilon\in[0,1]}\mathbb{E}\Big(\sup_{t\in[0,T]}\|Z_\varepsilon(t,\cdot)\|_m^p\Big)\leq C_p\sup_{\varepsilon\in\{0,1\}}\mathbb{E}\Big(\|Z_{\varepsilon,0}\|_m^p+K_m(T)^{p/2}\Big)<\infty. \tag{2.27}$$

Following the arguments in [10], this enables us to estimate moments of Z_1-Z_0 in terms of a process ζ_ε which is a formal derivative of Z_ε with respect to ε. Given operators or processes f_ε, $\varepsilon\in\{0,1\}$, set $f'=f_1-f_0$.

Theorem 2. *Let $m\geq 3$, and $p\in[2,\infty)$. Then for any integer $\kappa=0,\cdots,m-2$*

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\|Z_1(t)-Z_0(t)\|_\kappa^p\Big)\leq\sup_{\varepsilon\in[0,1]}\mathbb{E}\Big(\sup_{t\in[0,T]}\|\zeta_\varepsilon(t)\|_\kappa^p\Big), \tag{2.28}$$

where ζ_ε is the unique solution to the following linear evolution equation:

$$\begin{aligned}d\zeta_\varepsilon(t)=&\sum_{r=0}^{d_1}\big(L_{\varepsilon,r}\zeta_\varepsilon(t,x)+L'_rZ_\varepsilon(t,x)+F'_r(t,x)\big)dV_r(t)\\&+\sum_{l\geq 1}\big(S_{\varepsilon,l}\zeta_\varepsilon(t,x)+S'_lZ_\varepsilon(t,x)+G'_l(t,x)\big)dM_t^l\,,\end{aligned} \tag{2.29}$$

with the initial condition $Z'_0=Z_1-Z_0$. Furthermore,

$$\sup_{\varepsilon\in[0,1]}\mathbb{E}\Big(\sup_{t\in[0,T]}\|\zeta_\varepsilon(t)\|_{m-2}^p\Big)<\infty. \tag{2.30}$$

Proof. Using (2.27) we deduce that the processes $\tilde{F}_r(t,x) = L'_r Z_\varepsilon(t,x) + F'_r(t,x)$ and $\tilde{S}_l(t,x) = S'_l Z_\varepsilon(t,x) + G'_l(t,x)$ satisfy the assumption **(A4(m-2,p))** with $m-2 \geq 1$. Hence the existence and uniqueness of the process ζ_ε, solution to (2.29), as well as (2.30) can be deduced from Theorem 1.

We now prove (2.28) for $\kappa \in \{0, \cdots, m-2\}$ and assume that the right hand-side is finite. Given $(f_\varepsilon, \varepsilon \in [0,1))$, for and $h>0$ and $\varepsilon \in [0,1]$ such that $\varepsilon + h \in [0,1]$, set $\delta_h f_\varepsilon = (f_{\varepsilon+h} - f_\varepsilon)/h$. We at first prove that (2.28) can be deduced from the following: for every $\varepsilon \in [0,1)$, as $h \to 0$ is such that $h + \varepsilon \in [0,1]$,

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|\delta_h Z_\varepsilon(t) - \zeta_\varepsilon(t)\|_0^p \Big) \to 0. \tag{2.31}$$

Indeed, assume that (2.31) holds and for $n>0$ let $R_n = n^\kappa \Delta^\kappa (n\mathrm{Id} - \Delta)^{-\kappa}$ denote the *kapa*-fold composition of the resolvent of the Laplace operator Δ on the space $L^2 = H^0$. Then, by some classical estimates, there exists a constant $C(\kappa) > 0$ such that for any $\phi \in L^2$, $\|R_n h\|_\kappa \leq C(\kappa)\|\phi\|_0$. Hence (2.31) yields that for every $n>0$, as $h \to 0$ with $\varepsilon + h \in [0,1]$, we have $\mathbb{E}\Big(\sup_{t\in[0,T]} \|\delta_h R_n Z_\varepsilon(t) - R_n \zeta_\varepsilon(t)\|_\kappa^p \Big) \to 0$. Furthermore, since for every integer $N \geq 1$, we have $Z_1 - Z_0 = \frac{1}{N}\sum_{k=0}^{N-1} \delta_{1/N} Z_{k/N} \leq \sup_{\varepsilon\in[0,1]} \delta_{1/N} Z_\varepsilon$, we deduce that for every $n>0$ and $p \in [2,\infty)$:

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|R_n Z_0(t) - R_n Z_1(t)\|_\kappa^p \Big) \leq \sup_{\varepsilon\in[0,1]} \mathbb{E}\Big(\sup_{t\in[0,T]} \|R_n \zeta_\varepsilon(t)\|_\kappa^p \Big).$$

Finally, if $\phi \in H^0$ is such that $\liminf_{n\to\infty} \|R_n\phi\|_\kappa = N_\kappa < \infty$, then $\phi \in H^\kappa$ and $\|\phi\|_\kappa \leq N_\kappa$. Thus, by applying the Fatou Lemma and using estimate (2.30) we can conclude the proof of (2.28).

We will now prove the convergence (2.31). It is easy to see that the process $\eta_{\varepsilon,h}(t,\cdot) := \delta_h Z_\varepsilon(t,\cdot) - \zeta_\varepsilon(t,\cdot)$ has initial condition $\eta_{\varepsilon,h}(0) = 0$, and is a solution of the evolution equation:

$$\begin{aligned} d\eta_{\varepsilon,h}(t) = & \sum_{r=0}^{d_1} \big[L_{\varepsilon,r}\eta_{\varepsilon,h}(t,\cdot) + L'_r\big(Z_{\varepsilon+h}(t,\cdot) - Z_\varepsilon(t,\cdot)\big)\big] dV_t^r \\ & + \sum_{l\geq 1} \big[S_{\varepsilon,l}\eta_{\varepsilon,h}(t,\cdot) + S'_l\big(Z_{\varepsilon+h}(t,\cdot) - Z_\varepsilon(t,\cdot)\big)\big] dM_t^l. \end{aligned}$$

Hence, using once more Theorem 1, we deduce the existence of a constant $C_p > 0$ independent of $\varepsilon \in [0,1)$ and $h > 0$, such that $\varepsilon + h \in [0,1]$,

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|\delta_h Z_\varepsilon(t) - \zeta_\varepsilon(t)\|_0^p \Big) \leq C_p \mathbb{E}\Big(\int_0^T \|Z_{\varepsilon+h}(t) - Z_\varepsilon(t)\|_2^2 \, dV_t \Big)^{p/2}.$$

Using the interpolation inequality $\|\phi\|_2 \leq C\|\phi\|_0^{1/3}\|\phi\|_3^{2/3}$, see for instance Proposition 2.3 in [17], the Hölder inequality and the estimate (2.27) with $m=3$ from

Corollary 1, we deduce that

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\|\delta_h Z_\varepsilon(t)-\zeta_\varepsilon(t)\|_0^p\Big)\le C\Big[\mathbb{E}\Big(\sup_{t\in[0,T]}\|Z_{\varepsilon+h}(t)-Z_\varepsilon(t)\|_0^p\Big)\Big]^{1/3}.$$

Finally, the process $\Phi_{\varepsilon,h}(t,\cdot)=Z_{\varepsilon+h}(t,\cdot)-Z_\varepsilon(t,\cdot)$ is solution to the evolution equation

$$\begin{aligned}d\Phi_{\varepsilon,h}(t)=&\sum_{r=0}^{d_1}\big[L_{\varepsilon,r}\Phi_{\varepsilon,h}(t,\cdot)+hL'_rZ_{\varepsilon+h}(t,\cdot)+hF'_r(t,\cdot)\big]dV_t^r\\&+\sum_{l\ge1}\big[S_{\varepsilon,l}\Phi_{\varepsilon,h}(t,\cdot)+hS'_lZ_{\varepsilon+h}(t,\cdot)+hG'_l(t,\cdot))\big]dM_t^l,\end{aligned}$$

with the initial condition $\Phi_{\varepsilon,h}(0)=h(Z_1-Z_0)$. Thus, (2.27) and Theorem 1 prove the existence of a constant C, which does not depend on $\varepsilon\in[0,1)$ and $h>0$ with $\varepsilon+h\in[0,1]$, and such that

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\|Z_{\varepsilon+h}(t)-Z_\varepsilon(t)\|_0^p\Big)\le Ch^{p/3}.$$

This concludes the proof of (2.31) and hence that of the Theorem 2. □ □

3. Speed of convergence

3.1. **Convergence for time-independent coefficients.** For $r=0,\cdots,d_1$, $\varepsilon=0,1$, let $(V^r_{\varepsilon,t},t\in[0,T])$ be increasing processes which satisfy Assumptions **(A1)**, **(A2)**, **(A3(m+3))** and **(A4(m+3,p))** for some integer $m\ge1$, some $p\in[2,+\infty)$ separately for the increasing processes $(V^r_{\varepsilon,t},t\in[0,T])$, the same increasing process $(V_t,t\in[0,T])$ and the initial conditions $Z_{\varepsilon,0}$, $\varepsilon=0,1$. For $\varepsilon=0,1$, let Z_ε denote the solution to the evolution equation

$$dZ_\varepsilon(t,x)=\sum_{0\le r\le d_1}\big[L_rZ_\varepsilon(t,x)+F_r(x)\big]dV^r_{\varepsilon,t}+\sum_{l\ge1}\big[S_lZ_\varepsilon(t,x)+G_l(x)\big]dM_t^l,\quad(3.1)$$

with the initial conditions $Z_0(0,\cdot)=Z_{0,0}$ and $Z_1(0,\cdot)=Z_{1,0}$ respectively. Let

$$A:=\sup_{\omega\in\Omega}\sup_{t\in[0,T]}\max_{r=0,1,\cdots,d_1}|V^r_{1,t}-V^r_{0,t}|.$$

Then the H^m norm of the difference Z_1-Z_0 can be estimated in terms of A as follows when the coefficients of L_r and F_r are time-independent. Indeed, unlike the statements in [12], but as it is clear from the proof, the diffusion coefficients σ_l and G_l can depend on time.

Theorem 3. *Let L_r and F_r be time-independent, $\mathcal{F}_0$-measurable, V^r_ε, $\varepsilon=0,1$, M_l be as above and let Assumptions* **(A1)**, **(A2)**, **(A3(m+3))** *and* **(A4(m+3,p))** *be satisfied for some $m\ge0$ and some $p\in[2,+\infty)$. Suppose furthermore that*

$$\mathbb{E}\Big(\Big|\sum_{r=0}^{d_1}\|F_r\|^2_{m+1}\Big|^{p/2}+\sup_{s\in[0,T]}\Big|\sum_{l\ge1}\|G_r(s)\|^2_{m+2}\Big|^{p/2}\Big)<\infty.\quad(3.2)$$

Then there exists a constant $C>0$, which only depends on d and the constants in the above assumptions, such that the solutions Z_0 and Z_1 to (3.1) satisfy the following inequality:

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\|Z_1(t)-Z_0(t)\|_m^p\Big)\le C\,\Big(\mathbb{E}(\|Z_{1,0}-Z_{0,0}\|_m^p)+A^p\Big).$$

The proof of Theorem 3 will require several steps. Some of them do not depend on the fact that the coefficients are time independent; we are keeping general coefficients whenever this is possible. The first step is to use Theorem 2 and hence to define a process Z_ε for any $\varepsilon\in[0,1]$; it does not depend on the fact that the coefficients are time-independent and extends to the setting of the previous section. For $\varepsilon\in[0,1]$, $r=0,\cdots d_1$, $t\in[0,T]$ and $x\in\mathbb{R}^d$, let

$$V^r_{\varepsilon,t}=\varepsilon V^r_{1,t}+(1-\varepsilon)V^r_{0,t},\quad \rho^r_{\varepsilon,t}=dV^r_{\varepsilon,t}/dV_t$$

and for $j,k=1,\cdots d$, set $a^{j,k}_{\varepsilon,r}(t,x)=\rho^r_{\varepsilon,t}a^{j,k}_r(t,x)$, $b^{j,k}_{\varepsilon,r}(t)=\rho^r_{\varepsilon,t}b^{j,k}_r(t)$, $a^j_{\varepsilon,r}(t,x)=\rho^r_{\varepsilon,t}a^j_r(t,x)$, $a_{\varepsilon,r}(t,x)=\rho^r_{\varepsilon,t}a_r(t,x)$, $b_{\varepsilon,r}(t,x)=\rho^r_{\varepsilon,t}b_r(t,x)$, $L_{\varepsilon,r}=\rho^r_{\varepsilon,t}L_r$, $F_{\varepsilon,r}(t,x)=\rho^r_{\varepsilon,t}F_r(t,x)$. Then for $\varepsilon\in[0,1]$, the solution $Z_\varepsilon(t,\cdot)$ to equation (2.1) with the increasing processes $V^r_{\varepsilon,t}$ can be rewritten as (2.26) with the initial data $Z_\varepsilon(0)=\varepsilon Z_{1,0}+(1-\varepsilon)Z_{0,0}$ and the operators (resp. processes) $S_{\varepsilon,l}=S_l$ (resp. $G_{\varepsilon,l}=G_l$). Furthermore, we have

$$\sum_{r=0}^{d_1}\sum_{j,k=1}^{d}\lambda^j\lambda^k\big(a^{j,k}_{\varepsilon,r}(t,x)+ib^{j,k}_{\varepsilon}(t)\big)dV^r_t=$$
$$\sum_{0\le r\le d_1}\sum_{j,k=1}^{d}\lambda^j\lambda^k\big(a^{j,k}_r(t,x)+ib^{j,k}_r(t)\big)dV^r_{\varepsilon,t}.$$

Hence the conditions **(A1), (A2)**, **(A3(m+3))** and **(A4(m+3,p))** are satisfied. Therefore, using Theorem 2, one deduces that the proof of Theorem 3 reduces to check that

$$\sup_{\varepsilon\in[0,1]}\mathbb{E}\Big(\sup_{t\in[0,1]}\|\zeta_\varepsilon(t)\|_m^p\Big)\le C\big(\mathbb{E}\|Z_{1,0}-Z_{0,0}\|_m^p+A^p\big),\tag{3.3}$$

where if one lets $A^r_t=V^r_{1,t}-V^r_{0,t}$, the process ζ_ε is the unique solution to (2.29) which here can be written as follows: for $t\in[0,T]$ one has

$$\begin{aligned} d\zeta_\varepsilon(t) &= \sum_{r=0}^{d_1}\big[L_r\zeta_\varepsilon(t,x)dV^r_{\varepsilon,t}+\big(L_rZ_\varepsilon(t,x)+F_r(t,x)\big)dA^r_t\big]\\ &\quad+\sum_{l\ge1}S_l\zeta_\varepsilon(t,x)dM_l(t), \end{aligned}\tag{3.4}$$

and the initial condition is $\zeta_\varepsilon(0)=Z_{1,0}-Z_{0,0}$.
To ease notations, given a multi-index α, $j,k\in\{1,\cdots,d\}$ and Z smooth enough, set $Z_\alpha=D^\alpha Z$, $\quad Z_{\alpha,j}=D^\alpha D_jZ\quad$ and $Z_{\alpha,j,k}=D^\alpha D_jD_kZ$, so that for $Z,\zeta\in H^m$,

$(Z,\zeta)_m = \sum_{|\alpha\leq m} (Z_\alpha, \zeta_\alpha)_0$. Let

$$\mathcal{A} = \Big\{ \sum_\alpha \sum_\beta a^{\alpha,\beta} Z_\alpha Z_\beta \; ; a^{\alpha,\beta} \text{ uniformly bounded and complex-valued, } Z \in H^{m+3} \Big\}$$

and for $\Phi, \Psi \in \mathcal{A}$ set $\Phi \sim \Psi$ if there exists $Z \in H^m$ such that $\int_{\mathbb{R}^d} (\Phi - \Psi)(x) dx = \int_{\mathbb{R}^d} \Gamma(x) dx$, where Γ is a function defined by

$$\Gamma(x) = \sum_{|\alpha|\leq m} Z_\alpha(x) \overline{P^\alpha Z(x)} \quad \text{with} \quad P^\alpha Z = \sum_{|\beta|\leq m} \gamma^{\alpha,\beta} Z_\beta, \tag{3.5}$$

for some complex-valued functions $\gamma^{\alpha,\beta}$ such that $|\gamma^{\alpha,\beta}|$ are estimated from above by the constants appearing in Assumptions **(A1), (A2)**, **(A3(m+3))**, **(A4(m+3,p))**. Note that if Γ is as above, then for some constant $C_m(\Gamma)$ we have

$$\int_{\mathbb{R}^d} |\Gamma(x)| \, dx \leq C_m(\Gamma) \|Z\|_m^2. \tag{3.6}$$

For $\varepsilon > 0$, $j,k = 1, \cdots d$, $l \geq 1$ and $t \in [0,T]$, set $q_t^l = d\langle M^l \rangle_t / dV_t$ and let

$$\tilde{a}_\varepsilon^{j,k} := \tilde{a}_\varepsilon^{j,k}(t,\cdot) = \sum_{r=0}^{d_1} a_{\varepsilon,r}^{j,k}(t,\cdot) - \frac{1}{2} \sum_{l\geq 1} \sigma_l^j(t,\cdot) \sigma_l^k(t,\cdot) \, q_t^l.$$

For $m \geq 0$ and $z \in H^{m+1}$, set

$$[Z]_m^2 := [Z]_m^2(t) = \sum_{j,k=1}^d \big(\tilde{a}_\varepsilon^{j,k}(t) D_j Z \,,\, D_k Z \big)_m + C_m \|Z\|_m^2, \tag{3.7}$$

with $C_0 = 0$ and $C_m > 0$ to be chosen later on, so that the right handside of (3.7) is non negative. Given $Z, \zeta \in H^{m+1}$, set $[Z,\zeta]_m = \frac{1}{4}\big([Z+\zeta]_m^2 + [Z-\zeta]_m^2 \big)$. We at first prove that $[.]_m$ defines a non-negative quadratic form on H^{m+1} for some large enough constant C_m. Once more, this result does not require that the coefficients be time-independent.

Lemma 1. *Suppose that the conditions* **(A1), (A2)** *and* **(A3(m+1))** *are satisfied. Then there exists a large enough constant C_m such that* (3.7) *defines a non-negative quadratic form on H^{m+1}.*

Proof. Assumption **(A2)** for $\varepsilon \in \{0,1\}$ implies that (3.7) holds for $m = 0$ and $C_0 = 0$. Let $m \geq 1$ and α be a multi-index such that $1 \leq |\alpha| \leq m$. The Leibnitz formula implies the existence of constants $C(\alpha,\beta,\gamma)$ such that for $Z \in H^{m+1}$,

$$\Big(\sum_{j,k=1}^d \big(\tilde{a}_\varepsilon^{j,k}(t) D_j Z \big)_\alpha \,,\, Z_{\alpha,k} \Big)_0 =$$

$$\sum_{j,k=1}^d \big(\tilde{a}_\varepsilon^{j,k}(t) Z_{\alpha,j} \,,\, Z_{\alpha,k} \big)_0 + \sum_{\beta+\gamma=\alpha, |\beta|\geq 1} C(\alpha,\beta,\gamma) I_\varepsilon^{\alpha,\beta,\gamma}(t),$$

where $I_\varepsilon^{\alpha,\beta,\gamma}(t) := \sum_{j,k=1}^d \big(D^\beta \tilde{a}_\varepsilon^{j,k}(t) Z_{\gamma,j}\, ,\, Z_{\alpha,k}\big)_0$. Furthermore given $m\geq 1$, multi-indices α with $|\alpha|\leq m$ and $Z\in H^{m+1}$, using Assumption **(A2)** we deduce that $\sum_{1\leq j,k\leq d} \big(\tilde{a}_\varepsilon^{j,k}(t) Z_{\alpha,j}\, ,\, Z_{\alpha,k}\big)_0 \geq 0$. Thus, the proof reduces to check that

$$I_\varepsilon^{\alpha,\beta,\gamma}(t) \sim 0. \tag{3.8}$$

Indeed, then the upper estimate (3.6) proves (3.7), which concludes the proof of the Lemma. Integration by parts implies $I_\varepsilon^{\alpha,\beta,\gamma}(t) = -\sum_{j,k=1}^d \Big(D_k\big(D^\beta \tilde{a}_\varepsilon^{j,k}(t) Z_{\gamma,j}\big), Z_\alpha\Big)_0$. Since $|\beta|\leq m$, by Assumption **(A3(m+1))** we know that $D_k D^\beta \tilde{a}_\varepsilon^{j,k}(t)$ is bounded and hence $I_\varepsilon^{\alpha,\beta,\gamma}(t) \sim -\sum_{j,k=1}^d \big(D^\beta \tilde{a}_\varepsilon^{j,k}(t) Z_{\gamma,j,k}\, ,\, Z_\alpha\big)_0$. If $|\beta|\geq 2$, then $|\gamma|\leq m-2$; hence by **(A3(m))** we deduce that $I_\varepsilon^{\alpha,\beta,\gamma}(t)\sim 0$. If $|\beta|=1$, then $\tilde{a}_\varepsilon^{j,k}(t) = \tilde{a}_\varepsilon^{k,j}(t)$, so that

$$\begin{aligned} I_\varepsilon^{\alpha,\beta,\gamma}(t) &= \sum_{j,k=1}^d \big(D^\beta \tilde{a}_\varepsilon^{j,k}(t) Z_{\gamma,j}\, ,\, D^\beta Z_{\gamma,k}\big)_0 \\ &\sim \frac{1}{2}\sum_{j,k=1}^d \int_{\mathbb{R}^d} D^\beta \tilde{a}_\varepsilon^{j,k}(t,x) D^\beta\big(X_{\gamma,j}(.)X_{\gamma,k}(.) + Y_{\gamma,j}(.)Y_{\gamma,k}(.)\big)(x)dx. \end{aligned}$$

Thus, integrating by parts and using **(A3(2))** and the inequality $|\gamma|+1\leq m$, we deduce that $I_\varepsilon^{\alpha,\beta,\gamma}(t) \sim -\frac{1}{2}\sum_{j,k=1}^d \big(D^\beta D^\beta \tilde{a}_\varepsilon^{j,k}(t) Z_{\gamma,j}\, ,\, Z_{\gamma,k}\big)_0 \sim 0$. This concludes the proof. $\square$

The following lemma gathers some technical results which again hold for time-dependent coefficients.

Lemma 2. *Suppose that the assumptions of Theorem 3 hold. There exists a constant C such that for $\zeta\in H^{m+1}$ one has dV_t a.e.*

$$p(\zeta) := 2\sum_{0\leq r\leq d_1} \rho_{\varepsilon,t}^r \langle \zeta, L_r\zeta\rangle_m + \sum_{l\geq 1} q_t^l \|S_l\zeta\|_m^2 + 2[\zeta]_m^2 \leq C\|\zeta\|_m^2. \tag{3.9}$$

For any $\tilde{r}=0,\,1,\,\cdots,\,d_1$, $Z\in H^{m+3}$ and $\zeta\in H^{m+1}$ let

$$q_{\tilde{r}}(\zeta, Z) = \sum_{0\leq r\leq d_1} \rho_{\varepsilon,t}^r \big[\langle L_r\zeta, L_{\tilde{r}} Z\rangle_m + \langle \zeta, L_{\tilde{r}} L_r Z\rangle_m\big] + \sum_{l\geq 1} q_t^l (S_l\zeta, L_{\tilde{r}} S_l Z)_m.$$

Then there exists a constant C such that for any $Z\in H^{m+3}$ and $\zeta\in H^{m+1}$, one has dV_t a.e.

$$|q_{\tilde{r}}(\zeta, Z)| \leq C\|Z\|_{m+3}\big(\|\zeta\|_m + [\zeta]_m\big). \tag{3.10}$$

Proof. Suppose at first that $\zeta\in H^{m+2}$; since the upper estimates (3.9) and (3.10) only involve the H^{m+1}-norm of ζ, they will follow by approximation. Then we have

$$\sum_{0\leq r\leq d_1} 2\rho_{\varepsilon,t}^r \,\langle \zeta, L_r\zeta\rangle_m + \sum_{l\geq 1} q_t^l \,\|S_l\zeta\|_m^2 = \sum_{|\alpha|\leq m} Q_t^\alpha(\zeta,\zeta),$$

where

$$Q_t^\alpha(\zeta,\zeta) = 2\sum_{0\le r\le d_1} \rho^r_{\varepsilon,t}\,\big(\zeta_\alpha\,,\,(L_r\zeta)_\alpha\big)_0 + \sum_{l\ge 1} q_t^l\,\|(S_l\zeta)_\alpha\|_0^2.$$

Integration by parts and assumption **(A3(m))** imply that for $|\alpha|\le m$, one has

$$\begin{aligned}
2\big(\zeta_\alpha\,,\,(a^j_{\varepsilon,r}\zeta_j)_\alpha\big)_0 &\sim 2\int_{\mathbb{R}^d} a^j_{\varepsilon,r}(t,x)\big(X_\alpha(x)X_{\alpha,j}(x)+Y_\alpha(x)Y_{\alpha,j}(x)\big)dx\\
&= \int_{\mathbb{R}^d} a^j_{\varepsilon,r}(t,x)\big(X_\alpha^2+Y_\alpha^2\big)_j(x)\,dx \sim -\big(a^j_{\varepsilon,r}(t)_j\,\zeta_\alpha\,,\,\zeta_\alpha\big)_0\sim 0,\\
&\big(\zeta_\alpha\,,\,\big((a_{\varepsilon,r}+ib_{\varepsilon,r})\zeta\big)_\alpha\big)_0 \sim 0,\\
2\Big(\big(\sigma_l^j\zeta_j\big)_\alpha\,,\,\big((\sigma_l+i\tau_l)\zeta\big)_\alpha\Big)_0 &\sim 2\Big(\sigma_l^j\,\zeta_{\alpha,j}\,,\,(\sigma_l+i\tau_l)\zeta_\alpha\Big)_0\\
&\sim -\int_{\mathbb{R}^d}\big(\sigma_l^j\sigma_l\big)\big)_j(x)\,|\zeta_\alpha(x)|^2\,dx\sim 0,\\
&\big\|\big((\sigma_k+i\tau_k)\zeta\big)_\alpha\big\|_0^2\sim 0.
\end{aligned}$$

Finally, we have $\Big(\zeta_\alpha\,,\,\sum_{j,k}\big(D_k\big(ib_r^{j,k}(t)D_j\zeta\big)_\alpha\Big)_0=0$. Set $L_r^0\zeta=\sum_{j,k=1}^d D_k\big(a_r^{j,k}D_j\zeta\big)$ and $S_l^0\zeta=\sum_{j=1}^d\big(\sigma_l^j+i\tau_l^j\big)D_j\zeta$. Then we have

$$Q_t^\alpha(\zeta,\zeta)\sim 2\sum_{r=0}^{d_1}\rho^r_{\varepsilon,t}\big(\zeta_\alpha\,,\,(L_r^0\zeta)_\alpha\big)_0+\sum_{l\ge1}q_t^l\|\big(S_l^0\zeta\big)_\alpha\|_0^2. \tag{3.11}$$

If $m=0$, integration by parts proves that the right hand side of (3.11) is equal to $-2[\zeta]_0^2$ (with $C_0=0$). Let $m\ge 1$ and α be a multi index such that $m\ge|\alpha|\ge1$; set $\Gamma(\alpha)=\{(\beta,\gamma):\alpha=\beta+\gamma,|\beta|=1\}$. For $\phi,\psi\in H^m$, let $C(\beta,\gamma)$ be coefficients such that:

$$D^\alpha(\phi\psi)=\phi D^\alpha\psi+\sum_{(\beta,\gamma)\in\Gamma(\alpha)}C(\beta,\gamma)D^\beta\phi\,D^\gamma\psi+\sum_{\beta+\gamma=\alpha,|\beta|\ge2}C(\beta,\gamma)D^\beta\phi\,D^\gamma\psi.$$

This yields

$$\begin{aligned}
\sum_{l\ge1}q_t^l\big\|\big(S_l^0\zeta\big)_\alpha\big\|_0^2 &\sim \sum_{l\ge1}q_t^l\sum_{j,k=1}^d\Big\{\big(\sigma_l^k\,\zeta_{\alpha,k}\,,\,\sigma_l^j\,\zeta_{\alpha,j}\big)_0\\
&\qquad +2\sum_{(\beta,\gamma)\in\Gamma(\alpha)}C(\beta,\gamma)\big(D^\beta\sigma_l^k\zeta_{\gamma,k}\,,\,\sigma_l^j\zeta_{\alpha,j}\big)_0\Big\}.\\
&\sim\sum_{l\ge1}q_t^l\sum_{j,k=1}^d\Big\{\Big(\sigma_l^k\sigma_l^j\,\zeta_{\alpha,k}\,,\,\zeta_{\alpha,j}\Big)_0+2C(\alpha,\beta)\Big(D^\beta\sigma_l^k\sigma_l^j\,\zeta_{\gamma,k}\,,\,\zeta_{\alpha,j}\Big)_0.
\end{aligned}$$

Since for $(\beta,\gamma)\in\Gamma(\alpha)$ we have $|\gamma|+1\le|\alpha|\le m$ while $|\beta|+1=2$, integrating by parts and using **(A3(m))** we have for fixed l,

$$2q_t^l\sum_{j,k}\Big(D^\beta\sigma_l^k\sigma_l^j\zeta_{\gamma,k}\,,\,\zeta_{\alpha,j}\Big)_0=-q_t^l\sum_{j,k}\big(D^\beta(\sigma_l^k\sigma_l^j)\zeta_{\gamma,j,k}\,,\,\zeta_\alpha\big)_0.$$

Furthermore, integration by parts and **(A3(m))** yield

$$\begin{aligned}2\rho^r_{\varepsilon,t}\big(\zeta_\alpha\,,\,(L^0\zeta)_\alpha\big)_0\sim&-2\sum_{j,k}\Big\{\big(a^{j,k}_{\varepsilon,r}\zeta_{\alpha,j}\,,\,\zeta_{\alpha,k}\big)_0\\&-\sum_{(\beta,\gamma)\in\Gamma(\alpha)}C(\beta,\gamma)\big(D^\beta(a^{j,k}_{\varepsilon,r})\zeta_{\gamma,j}\,,\,\zeta_{\alpha,k}\big)_0\Big\}.\end{aligned}$$

Therefore, the definition of $\tilde a^{j,k}_{\varepsilon,r}$, (3.11) and (3.8) yield

$$\begin{aligned}Q_t^\alpha(\zeta,\zeta)&\sim-2\sum_{j,k}\big(\tilde a^{j,k}_\varepsilon\,\zeta_{\alpha,j}\,,\,\zeta_{\alpha,k}\big)_0-2\sum_{j,k}\sum_{(\beta,\gamma)\in\Gamma(\alpha)}C(\beta,\gamma)\big(D^\beta(\tilde a^{j,k}_\varepsilon)\zeta_{\gamma,j}\,,\,\zeta_{\alpha,k}\big)_0\\&\sim-2\sum_{j,k}\Big(\big(\zeta_j\tilde a^{j,k}_\varepsilon\big)_\alpha\,,\,\zeta_{\alpha,k}\Big)_0.\end{aligned}$$

Hence for $\zeta\in H^{m+1}$,

$$p(\zeta)=\sum_{|\alpha|\le m}\int_{\mathbb{R}^d}Q_t^\alpha(\zeta,\zeta)dx+2[\zeta]_m^2=2\sum_{|\alpha|\le m}(\zeta_\alpha,P^\alpha\zeta)_0,\tag{3.12}$$

for some operator P^α which satisfies (3.5). Hence (3.6) concludes the proof of (3.9).

Polarizing (3.12), we deduce that for $\tilde Z,\zeta\in H^{m+1}$,

$$\begin{aligned}\sum_{r=0}^{d_1}\rho^r_{\varepsilon,t}\big[\langle\tilde Z,L_r\zeta\rangle_m+\langle L_r\tilde Z,\zeta\rangle_m\big]+\sum_{l\ge1}q_t^l\big(S_l\tilde Z,S_l\zeta\big)_m+2[\tilde Z,\zeta]_m\\=\sum_{|\alpha|\le m}\big[(\tilde Z_\alpha,P^\alpha\zeta)_0+(\zeta_\alpha,P^\alpha\tilde Z)_0\big].\end{aligned}$$

Let $\tilde r\in\{0,1,\cdots,d_1\}$ and for $Z\in H^{m+3}$, $\zeta\in H^{m+1}$, set $\tilde Z=L_{\tilde r}Z$; then if one sets

$$q_{\tilde r}(\zeta,Z):=\sum_{r=0}^{d_1}\rho^r_{\varepsilon,t}\big[\langle L_r\zeta,L_{\tilde r}Z\rangle_m+\langle\zeta,L_{\tilde r}L_rZ\rangle_m\big]+\sum_{l\ge1}q_t^l\,\big(S_l\zeta,L_{\tilde r}S_lZ\big)_m,$$

one deduces that

$$\begin{aligned}&q_{\tilde r}(\zeta,Z)+\sum_r\rho^r_{\varepsilon,t}\big(\zeta,[L_rL_{\tilde r}-L_{\tilde r}L_r]Z\big)_m+2\sum_l q_t^l\big(S_l\zeta,[S_lL_{\tilde r}-L_{\tilde r}S_l]Z\big)_m+2[\zeta,L_{\tilde r}Z]_m\\&=\sum_{|\alpha|\le m}\big[(D^\alpha L_{\tilde r}Z,P^\alpha\zeta)_0+(D^\alpha\zeta,P^\alpha L_{\tilde r}Z)_0\big].\end{aligned}$$

The operators $L_r L_{\tilde r} - L_{\tilde r} L_r$ and $S_l L_{\tilde r} - L_{\tilde r} S_l$ are of order 3 and 2 respectively. Hence integration by parts and the Cauchy Schwarz inequality imply that

$$|q_{\tilde r}(\zeta, Z)| \le C\|\zeta\|_m \|Z\|_{m+3} + C[\zeta]_m [Z]_m.$$

Finally, (3.7) and Assumption **(A3(m))** imply that for $Z \in H^{m+1}$,

$$[Z]_m^2 \le C\|Z\|_{m+1}^2 + C_m \|Z\|_m^2 \le C\|Z\|_{m+1}^2.$$

This concludes the proof of (3.10). □

The following lemma is based on some time integration by parts and requires the coefficients of L_r and F_r to be time independent.

Lemma 3. *Let the assumptions of Theorem 3 be satisfied and Z_ε (resp. ζ_ε) denote the processes defined by* (3.1) *(resp.* (3.4)*). For $r = 0, \cdots, d_1$ and $t \in [0,T]$, let $A^r_t = V^r_{1,t} - V^r_{0,t}$ and set*

$$J_{\varepsilon,t} := \sum_{r=0}^{d_1} \int_0^t \big(\zeta_\varepsilon(s), L_r Z_\varepsilon(s) + F_r\big)_m dA^r_s. \tag{3.13}$$

Then there exists a constant C such that for any stopping time $\tau \le T$,

$$\mathbb{E}\Big[\sup_{t\in[0,\tau]} \Big(J_{\varepsilon,t} - \int_0^t [\zeta_\varepsilon(s)]_m^2 dV_s\Big)_+^{p/2}\Big] \le \frac{1}{4p}\mathbb{E}\Big(\sup_{t\in[0,\tau]} \|\zeta_\varepsilon(s)\|_m^2\Big) + C\Big(A^p + \mathbb{E}\int_0^\tau \|\zeta_\varepsilon(s)\|_m^p\, dV_s\Big). \tag{3.14}$$

Proof. The main problem is to upper estimate $J_{\varepsilon,t}$ in terms of A and not in terms of the total variation of the measures dA^r_t. This requires some integration by parts; equations (3.1) and (3.4) imply:

$$J_{\varepsilon,t} = \sum_{r=0}^{d_1} \big(\zeta_\varepsilon(t)\,,\, L_r Z_\varepsilon(t) + F_r\big)_m A^r_t - \sum_{1\le k\le 4} J^k_{\varepsilon,t}, \tag{3.15}$$

where for $t \in [0,T]$ we set:

$$J^1_{\varepsilon,t} = \sum_r A^r_s \Big[\sum_{\tilde r} \langle L_{\tilde r}\zeta_\varepsilon(s), L_r Z_\varepsilon(s) + F_r\rangle_m + \langle \zeta_\varepsilon(s), L_r[L_{\tilde r} Z_\varepsilon(s) + F_{\tilde r}]\rangle_m\Big] dV^r_{\varepsilon,s},$$

$$J^2_{\varepsilon,t} = \int_0^t \sum_r A^r_s \sum_{l\ge 1} \big(S_l(s)\zeta_\varepsilon(s), L_r[S_l(s)Z_\varepsilon(s) + G_l(s)]\big)_m d\langle M^l\rangle_s,$$

$$J^3_{\varepsilon,t} = \int_0^t \sum_r A^r_s \sum_{l\ge 1} \big[\big(S_l(s)\zeta_\varepsilon(s), L_r Z_\varepsilon(s) + F_r\big)_m + \big(\zeta_\varepsilon(s), L_r[S_l(s)Z_\varepsilon(s) + G_l(s)]\big)_m\big] dM^l_s,$$

$$J^4_{\varepsilon,t} = \int_0^t \sum_r A^r_s \Big[\sum_{\tilde r} \big(L_{\tilde r} Z_\varepsilon(s) + F_{\tilde r}, L_r Z_\varepsilon(s) + F_r\big)_m\Big] dA^{\tilde r}_s.$$

Note that

$$J^4_{\varepsilon,t} = \frac{1}{2}\int_0^t \sum_{r,\tilde{r}} \big(L_{\tilde{r}} Z_\varepsilon(s) + F_{\tilde{r}}, L_r Z_\varepsilon(s) + F_r\big)_m d(A^r_s A^{\tilde{r}}_s).$$

Using (3.10), integration by parts, Assumption **(A3(m))**, the Cauchy-Schwarz and Young inequalities, we deduce that

$$\begin{aligned}
J^1_{\varepsilon,t} + J^2_{\varepsilon,t} &\leq CA\int_0^t \Big[\|Z_\varepsilon(s)\|_{m+3}\big\{[\zeta_\varepsilon(s)]_m + \|\zeta_\varepsilon(s)\|_m\big\} + \sum_r \|\zeta_\varepsilon(s)\|_m\, \|F_r\|_{m+2}\Big]\, dV_s \\
&\quad + CA\int_0^t \sum_l \|\zeta_\varepsilon(s)\|_m\, \|G_l(s)\|_{m+3}\, d\langle M^l\rangle_s \\
&\leq \int_0^t \big([\zeta_\varepsilon(s)]^2_m + \|\zeta_\varepsilon(s)\|^2_m\big)\, dV_s \\
&\quad + CA^2\int_0^t \Big[\Big(\|Z_\varepsilon(s)\|^2_{m+3} + \sum_r \|F_r\|^2_{m+2}\Big) dV_s + \sum_{l\geq 1} \|G_l(s)\|^2_{m+3}\, d\langle M^l\rangle_s\Big] \\
&\leq \int_0^t \big([\zeta_\varepsilon(s)]^2_m + \|\zeta_\varepsilon(s)\|^2_m\big)\, dV_s + CA^2\Big(\int_0^t \|Z_\varepsilon(s)\|^2_{m+3}\, dV_s + K_{m+2}(t)\Big),
\end{aligned}$$

where the last inequality is deduced from Assumption **(A4(m+2,2))**.
The Cauchy Schwarz inequality, integration by parts and Assumption **(A3(m+1))** imply that for fixed $r = 0,\cdots,d_1$ and $l \geq 1$,

$$\begin{aligned}
&|\big(S_l(s)\zeta_\varepsilon(s), L_r Z_\varepsilon(s) + F_r\big)_m| + |\big(\zeta_\varepsilon(s), L_r[S_l(s)Z_\varepsilon(s) + G_l(s)]\big)_m| \\
&\qquad\qquad \leq C\|\zeta_\varepsilon(s)\|_m\big[\|Z_\varepsilon(s)\|_{m+3} + \|F_r\|_{m+1} + \|G_l(s)\|_{m+2}\big].
\end{aligned}$$

Therefore, the Burkholder Davies Gundy inequality and Assumption **(A1)** imply that for any stopping time $\tau \leq T$, we have:

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in[0,\tau]} |J^3_{\varepsilon,t}|^{p/2}\Big) \\
&\leq CA^{p/2}\mathbb{E}\Big(\int_0^\tau \|\zeta_\varepsilon(s)\|^2_m\Big[\|Z_\varepsilon(s)\|^2_{m+3} + \sum_{0\leq r\leq d_1} \|F_r\|^2_{m+1} \\
&\qquad + \sup_{s\in[0,T]} \sum_{l\geq 1} \|G_l(s)\|^2_{m+2}\Big] d\langle M^l\rangle_s\Big)^{p/4} \\
&\leq CA^{p/2}\mathbb{E}\Big[\Big(\sup_{s\in[0,\tau]} \|Z_\varepsilon(s)\|^{p/2}_{m+3} + \Big|\sum_{r=0}^{d_1} \|F_r\|^2_{m+1}\Big|^{p/4} + \sup_{s\in[0,T]} \Big|\sum_{l\geq 1} \|G_l(s)\|^2_{m+2}\Big|^{p/4}\Big) \\
&\qquad\qquad \times \Big(\int_0^\tau \|\zeta_\varepsilon(s)\|^2_m dV_s\Big)^{p/4}\Big]
\end{aligned}$$

$$\leq CA^p\,\mathbb{E}\Big[\Big(\sup_{s\in[0,T]}\|Z_\varepsilon(s)\|_{m+3}^p\Big)+\Big|\tilde K\sum_{r=0}^{d_1}\|F_r\|_{m+1}^2\Big|^{p/2}+\sup_{s\in[0,T]}\Big|\sum_{l\geq 1}\|G_l(s)\|_{m+2}^2\Big|^{p/2}\Big]$$
$$+\frac{1}{8p}\mathbb{E}\Big(\sup_{s\in[0,\tau]}\|\zeta_\varepsilon(s)\|_m^p\Big)+C\mathbb{E}\int_0^\tau\|\zeta(s)\|_m^p dV_s.$$

Using the condition (3.2) and Theorem 1 with $m+3$, we deduce yhat

$$\mathbb{E}\Big(\sup_{s\in[0,\tau]}|J^3_{\varepsilon,t}|^{p/2}\Big)\leq\frac{1}{8p}\mathbb{E}\Big(\sup_{t\in[0,\tau]}\|\zeta_\varepsilon(t)\|_m^p\Big)+C\mathbb{E}\int_0^\tau\|\zeta_\varepsilon(s)\|_m^p dV_s+CA^p.$$

Therefore,

$$\mathbb{E}\Big\{\sup_{t\in[0,\tau]}\Big(J_{\varepsilon,t}-\int_0^t[\zeta_\varepsilon(s)]_m^p(s)dV_s\Big)_+^{p/2}\Big\}\leq 1/(8p)\mathbb{E}\Big(\sup_{t\in[0,\tau]}\|\zeta_\varepsilon(t)\|_m^2\Big)$$
$$+C\mathbb{E}\int_0^\tau\|\zeta_\varepsilon(s)\|_m^p dV_s+CA^p+C\mathbb{E}\Big(\sup_{t\in[0,\tau]}|J^4_{\varepsilon,t}|^{p/2}\Big).$$

Integrating by parts we obtain

$$2J^4_{\varepsilon,t}=\sum_{r,\tilde r}\big(L_{\tilde r}Z_\varepsilon(t)+F_{\tilde r}\,,\,L_rZ_\varepsilon(t)+F_r\big)_m A^r_tA^{\tilde r}_t-\sum_{j=1}^3 J^{4,j}_{\varepsilon,t}\,,\tag{3.16}$$

where

$$J^{4,1}_{\varepsilon,t}=2\sum_{r,\tilde r}\sum_{\bar r}\int_0^t A^r_sA^{\tilde r}_s\big\langle L_{\tilde r}[L_{\bar r}Z_\varepsilon(s)+F_{\bar r}]\,,\,L_rZ_\varepsilon(s)+F_r\big\rangle_m dV^{\bar r}_{\varepsilon,s},$$
$$J^{4,2}_{\varepsilon,t}=2\sum_{r,\tilde r}\sum_{l\geq 1}\int_0^t A^r_sA^{\tilde r}_s\big(L_{\tilde r}[S_l(s)Z_\varepsilon(s)+G_l(s)]\,,\,L_rZ_\varepsilon(s)+F_r\big)_m dM^l_s,$$
$$J^{4,3}_{\varepsilon,t}=\sum_{r,\tilde r}\sum_{l\geq 1}\int_0^t A^r_sA^{\tilde r}_s\big(L_{\tilde r}[S_l(s)Z_\varepsilon(s)+G_l(s)]\,,\,L_r[S_l(s)Z_\varepsilon(s)+G_l(s)]\big)_m d\langle M^l\rangle_s.$$

Integration by part, Assumption **(A3(m+2))**, the Cauchy-Schwarz and Young inequalities yield

$$|\big\langle L_{\tilde r}[L_{\bar r}Z_\varepsilon(s)+F_{\bar r}]\,,\,L_rZ_\varepsilon(s)+F_r\big\rangle_m|\leq C\big[\|Z_\varepsilon(s)\|_{m+3}^2+\|F_{\bar r}\|_{m+2}^2+\|F_r\|_m^2\big],$$
$$|\big(L_{\tilde r}[S_l(s)Z_\varepsilon(s)+G_l(s)],L_rZ_\varepsilon(s)+F_r\big)_m|\leq C\big[\|Z_\varepsilon(s)\|_{m+3}^2+\|G_l(s)\|_{m+2}^2+\|F_r\|_m^2\big].$$

Hence, using Theorem 1, (3.2), Assumptions **(A1)** and **(A4(m+2))** we deduce

$$\mathbb{E}\Big(\sup_{t\in[0,\tau]}\Big|J^{4,1}_{\varepsilon,t}+J^{4,3}_{\varepsilon,t}\Big|^{p/2}\Big)\leq CA^p.$$

Finally, the Burkholder-Davies-Gundy inequality implies that

$$\mathbb{E}\Big(\sup_{t\in[0,\tau]}|J^{4,2}_{\varepsilon,t}|^{p/2}\Big)$$

$$\leq CA^p\mathbb{E}\Big|\int_0^\tau \big|\big(L_{\tilde r}[S_l(s)Z_\varepsilon(s)+G_l(s)]\,,\,L_rZ_\varepsilon(s)+F_r\big)_m\big|^2 d\langle M^l\rangle_s\Big|^{p/4}\leq CA^p.$$

Hence, $\mathbb{E}\big(\sup_{t\in[0,\tau]}|J^4_{\varepsilon,t}|^{p/2}\big)\leq CA^p$, which concludes the proof. □

Using Lemmas 1-3, we now prove Theorem 3 for time-independent coefficients. *Proof of Theorem 3* Apply the operator D^α to both sides of (3.4) and use the Itô formula for $\|D^\alpha\zeta_\varepsilon(t)\|_0^2$. This yields

$$\begin{aligned}d\|\zeta_\varepsilon(t)\|_m^2 =& 2\sum_{|\alpha|\leq m}\sum_r\big[\big\langle\zeta_\varepsilon(t),L_r\zeta_\varepsilon(t)\big\rangle_m\rho^r_{\varepsilon,t}dV_t+\big(\zeta_\varepsilon(t),L_rZ_\varepsilon(t)+F_r\big)_m dA^r_t\big]\\&+\sum_{|\alpha|\leq m}\sum_{l\geq1}\big[\|S_l(t)\zeta_\varepsilon(t)\|_m^2q^l_tdV_t+2\big(\zeta_\varepsilon(t),S_l(t)\zeta_\varepsilon(t)\big)_m dM^l_t\big],\end{aligned}$$

where $\langle Z,\zeta\rangle_m$ denotes the duality between H^{m-1} and H^{m+1} which extends the scalar product in H^m. Using (3.9) we deduce that

$$d\|\zeta_\varepsilon(t)\|_m^2\leq -2[\zeta_\varepsilon(t)]_m^2dV_t+C\|\zeta_\varepsilon(t)\|_m^2dV_t+2dJ_{\varepsilon,t}+2\sum_{l\geq1}\big(\zeta_\varepsilon(t),S_l(t)\zeta_\varepsilon(t)\big)_m dM^l_t,$$

where $J_{\varepsilon,t}$ is defined by (3.13). Using (2.25) we deduce that $\big|\big(\zeta_\varepsilon(t),S_l(t)\zeta_\varepsilon(t)\big)_m\big|\leq C\|\zeta_\varepsilon(t)\|_m^2$. Thus Lemma 3, the Burkholder Davies Gundy inequality and Assumption **(A1)** yield for any stopping time $\tau\leq T$

$$\begin{aligned}\mathbb{E}\Big(\sup_{t\in[0,\tau]}\|\zeta_\varepsilon(t)\|_m^p\Big)&\leq C\mathbb{E}\|Z_{1,0}-Z_{0,0}\|_m^p+p\mathbb{E}\Big(\sup_{t\in[0,\tau]}J_{\varepsilon,t}-\int_0^t[\zeta_\varepsilon(s)]_m^2dV_s\Big)_+^{p/2}\\&\quad+C_p\mathbb{E}\Big|\int_0^\tau\|\zeta_\varepsilon(s)\|_m^4dV_s\Big|^{p/4}+C_p\mathbb{E}\Big|\int_0^\tau\|\zeta_\varepsilon(s)\|_m^2dV_s\Big|^{p/2}\\&\leq C\mathbb{E}\|Z_{1,0}-Z_{0,0}\|_m^p+\frac14\mathbb{E}\Big(\sup_{t\in[0,\tau]}\|\zeta_\varepsilon(t)\|_m^p\Big)+C\Big(A^p+\mathbb{E}\int_0^\tau\|\zeta_\varepsilon(s)\|_m^pdV_s\Big)\\&\quad+C\mathbb{E}\Big[\sup_{t\in[0,\tau]}\|\zeta_\varepsilon(t)\|_m^{p/2}\Big(\int_0^\tau\|\zeta_\varepsilon(s)\|_m^2dV_s\Big)^{p/4}\Big]+C_p\mathbb{E}\int_0^\tau\|\zeta_\varepsilon(s)\|_m^pdV_s\\&\leq C\mathbb{E}\|Z_{1,0}-Z_{0,0}\|_m^p+CA^p+\frac12\mathbb{E}\Big(\sup_{t\in[0,\tau]}\|\zeta_\varepsilon(t)\|_m^p\Big)+C\mathbb{E}\Big(\int_0^\tau\|\zeta_\varepsilon(s)\|_m^pdV_s\Big),\end{aligned}$$

where the last upper estimate follows from the Young inequality.
Let $\tau_N=\inf\{t:\|\zeta_\varepsilon(t)\|_m^p\geq N\}\wedge T$; then the Gronwall Lemma implies that

$$\mathbb{E}\Big(\sup_{t\in[0,\tau_N]}\|\zeta_\varepsilon(t)\|_m^p\Big)\leq C\big(\mathbb{E}\|Z_{1,0}-Z_{0,0}\|_m^p+A^p\big).$$

Letting $N\to\infty$ concludes the proof. □

3.2. **Case of the time dependent coefficients.** In this section, we prove a convergence result similar to that in Theorem 3 when the coefficients of the operators depend on time. Integration by parts in Lemma 3 will give extra terms, which require more assumptions to be dealt with.

Assumption (A5(m)) There exists an integer number d_2, an $\big(\mathcal{F}_t\big)$-continuous martingale $N_t = (N_t^1, \cdots, N_t^{d_2})$ and, for each $\gamma = 0, \cdots, d_2$ a bounded predicable process $h_\gamma : \Omega \times (0,T] \times \mathbb{R}^d \to \mathbb{R}^N$ for some N depending on d and d_1 such that

$$\begin{aligned} h_\gamma(t,x) \;\; &:= \;\; (a^{j,k}_{\gamma,r}(t,x), b^{j,k}_{\gamma,r}(t), a^j_{\gamma,r}(t,x), a_{\gamma,r}(t,x), b_{\gamma,r}(t,x), F_{\gamma,r}(t,x); \\ &\quad\;\; 1 \le j,k \le d,\ 0 \le r \le d_1,\ 1 \le \gamma \le d_2), \end{aligned}$$

for some symmetric non negative matrices $(a^{j,k}_{\gamma,r}(t,x), j,k = 1, \cdots, d)$ and $(b^{j,k}_{\gamma,r}(t), j,k = 1, \cdots, d)$. Furthermore, we suppose that for every $\omega \in \Omega$ and $t \in [0,T]$, the maps $h_\gamma(t,\cdot)$ are of class $\mathcal{C}^{m+1}$ such that for some constant K we have $|D^\alpha h_\gamma(t,\cdot)| \le K$ for any multi-index α with $|\alpha| \le m+1$ and such that for $t \in [0,T]$,

$$\begin{aligned} \sum_{\gamma=1}^{d_2} d\langle N^\gamma\rangle_t \;\; &\le \;\; dV_t, \\ h(t,x) \;\; &= \;\; h(0,x) + \int_0^t h_0(s,x)dV_s + \sum_{\gamma=1}^{d_2} \int_0^t h_\gamma(s,x)dN^\gamma_s. \end{aligned}$$

For $\gamma = 0, \cdots, d_2$, $r = 0, \cdots, d_1$, let $L_{\gamma,r}$ be the time dependent differential operator defined by:

$$\begin{aligned} L_{\gamma,r}Z(t,x) \;\; &= \;\; \sum_{j,k=1}^d D_k\Big(\big[a^{j,k}_{\gamma,r}(t,x) + ib^{j,k}_{\gamma,r}(t)\big]D_jZ(t,x)\Big) + \sum_{j=1}^d a^j_{\gamma,r}(t,x)D_jZ(t,x) \\ &\quad + \big[a_{\gamma,r}(t,x) + ib_{\gamma,r}(t,x)\big]Z(t,x). \end{aligned}$$

For $r = 0, \cdots, d_1$, let

$$L_rZ(t,x) = L_r(0)Z(0,x) + \int_0^t L_{0,r}Z(s,x)dV_s + \sum_{\gamma=1}^{d_2} L_{\gamma,r}Z(s,x)dN^\gamma_s,$$

and $F_r(t,x) = F_r(0,x) + \int_0^t F_{0,r}(s,x)dV_s + \sum_{\gamma=1}^{d_2} F_{\gamma,r}(s,x)dN^\gamma_s$. We then have the following abstract convergence result which extends Theorem3.

Theorem 4. *Let Assumptions* **(A(1))**, **(A(2))**, **(A3(m+3))**, **(A4(m+3,p))** *and* **(A5(m))** *be satisfied and suppose that*

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\Big|\sum_{r=0}^{d_1}\|F_r(t)\|^2_{m+1}\Big|^{p/2} + \sup_{t\in[0,T]}\Big|\sum_{l\ge1}\|G_{m+2}(t)\|^2_{m+2}\Big|^{p/2} < \infty. \tag{3.17}$$

Then there exists some constant $C > 0$ such that

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\|Z_1(t) - Z_0(t)\|^p_m\Big) \le C\Big(\mathbb{E}\big(\|Z_1(0) - Z_0(0)\|^p_m\big) + A^p\Big). \tag{3.18}$$

Proof. Since Lemmas 1 and 2 did not depend on the fact that the coefficients are time-independent, only Lemma 3 has to be extended. For $t \in [0,T]$, let

$$J_{\varepsilon,t} = \sum_{r=0}^{d_1} \int_0^t \big(\zeta_\varepsilon(s)\,,\, L_r(s) Z_\varepsilon(s) + F_r(s)\big)_m dA^r_s.$$

Since $A_0^r = 0$ for $r = 0, \cdots, d_1$, the integration by parts formula (3.15) has to be replaced by

$$J_{\varepsilon,t} = \sum_{r=0}^{d_1} \big(\zeta_\varepsilon(t)\,,\, L_r(t) Z_\varepsilon(t) + F_r(t)\big)_m A^r_t - \sum_{k=1}^{7} J^k_{\varepsilon,t},\ t \in [0,T],$$

where the additional terms on the right hand-side are defined for $t \in [0,T]$ as follows:

$$J^5_{\varepsilon,t} = \sum_r A^r_s \big(\zeta_\varepsilon(s), L_{0,r} Z_\varepsilon(s) + F_{0,r}\big)_m dV_s,$$

$$J^6_{\varepsilon,t} = \int_0^t \sum_r A^r_s \sum_{\gamma=1}^{d_2} \big(\zeta_\varepsilon(s), L_{\gamma,r} Z_\varepsilon(s) + F_{\gamma,r}(s)\big)_m dN^\gamma_s,$$

$$J^7_{\varepsilon,t} = \int_0^t \sum_r A^r_s \sum_{l\geq 1} \sum_{\gamma=1}^{d_2} \big(S_l(s)\zeta_\varepsilon(s), L_{\gamma,r} Z^\varepsilon(s) + F_{\gamma,r}(s)\big)_m d\langle M^l, N^\gamma\rangle_s.$$

Arguments similar to those used in the proof of Lemma 3, using integration by parts and the regularity assumptions of the coefficients, prove that for $k = 5, 6$ there exists a constant $C > 0$ such that for any stopping time $tau \leq T$ we have:

$$\begin{aligned}\mathbb{E}\Big(\sup_{t\in[0,\tau]} |J^k_{\varepsilon,t}|^{p/2}\Big) &\leq C A^{p/2} \mathbb{E}\Big(\sup_{t\in[0,\tau]} \|\zeta_\varepsilon(t)\|_m^{p/2} \sup_{t\in[0,\tau]} \big(\|Z_\varepsilon(t)\|_{m+2} + C\big)^{p/2}\Big) \\ &\leq \frac{1}{24p} \mathbb{E}\Big(\sup_{t\in[0,\tau]} \|\zeta_\varepsilon(t)\|_m^p\Big) + C A^p,\end{aligned}$$

where the last inequality follows from the Young inequality. Furthermore, the Burkholder-Davies-Gundy inequality and the upper estimates of the quadratic variations of the martingales N^γ yield for every $\gamma = 1, \cdots, d_2$, $r = 0, \cdots, d_1$ and $\tau \in [0,T]$,

$$\begin{aligned}\mathbb{E}\Big(\sup_{t\in[0,\tau]} |J^7_{\varepsilon,t}|^{p/2}\Big) \leq& C\mathbb{E}\Big(\int_0^\tau \big|A^r_s\big(\zeta_\varepsilon(s), L_{\gamma,r} Z_\varepsilon(s) + F_{\gamma,r}(s)\big)_m\big|^2 dV_s\Big)^{p/4} \\ &\leq C A^{p/2} \mathbb{E}\Big(\sup_{t\in[0,\tau]} \|\zeta_\varepsilon(t)\|_m^{p/2} \sup_{t\in[0,\tau]} \big(\|Z_\varepsilon(t)\|_{m+2} + C\big)^{p/2}\Big).\end{aligned}$$

Hence, the proof will completed by extending the upper estimate (3.16) as follows:

$$2J^4_{\varepsilon,t} = \sum_{r,\tilde r} \big(L_{\tilde r} Z_\varepsilon(t) + F_{\tilde r}(t)\,,\, L_r Z_\varepsilon(t) + F_r(t)\big)_m A^r_t A^{\tilde r}_t - \sum_{j=1}^{7} J^{4,j}_{\varepsilon,t},$$

where for $j = 4, \cdots, 7$ we have:

$$J^{4,4}_{\varepsilon,t} = 2 \sum_{r,\tilde{r}} \int_0^t A^r_s A^{\tilde{r}}_s \big(L_{\tilde{r},0}(s) Z_\varepsilon(s) + F_{\tilde{r},0}(s)\, ,\, L_r(s) Z_\varepsilon(s) + F_r(s) \big)_m dV_s,$$

$$J^{4,5}_{\varepsilon,t} = \sum_{r,\tilde{r}} \sum_{\gamma,\tilde{\gamma}} \int_0^t A^r_s A^{\tilde{r}}_s \big(L_{\tilde{\gamma},\tilde{r}}(s) Z_\varepsilon(s) + F_{\tilde{\gamma},\tilde{r}}(s)\, ,\, L_{\gamma,r} Z_\varepsilon(s) + F_{\gamma,r}(s) \big)_m d\langle N^{\tilde{\gamma}}, N^\gamma \rangle_s,$$

$$J^{4,6}_{\varepsilon,t} = 2 \sum_{r,\tilde{r}} \sum_{\gamma} \sum_{l\geq 1} \int_0^t A^r_s A^{\tilde{r}}_s \big(L_{\gamma,\tilde{r}}(s) Z_\varepsilon(s) + F_{\gamma,\tilde{r}}(s)\, ,$$
$$L_r(s) [S_l(s) Z_\varepsilon(s) + G_l(s)] \big)_m d\langle N^\gamma, M^l \rangle_s,$$

$$J^{4,7}_{\varepsilon,t} = 2 \sum_{r,\tilde{r}} \sum_{\gamma} \int_0^t A^r_s A^{\tilde{r}}_s \big(L_{\gamma,\tilde{r}}(s) Z_\varepsilon(s) + F_{\gamma,\tilde{r}}(s)\, ,\, L_r(s) Z_\varepsilon(s) + F_r(s) \big)_m dN^\gamma_s.$$

We obtain upper estimates of the terms $\mathbb{E}\big(\sup_{t\in[0,\tau]} |J^{4,k}_{\varepsilon,t}|^{p/2}\big)$ for $k = 4, \cdots, 7$ by arguments similar to that used for $k = 1, \cdots, 3$, which implies $\mathbb{E}\big(\sup_{t\in[0,\tau]} |J^{4}_{\varepsilon,t}|^{p/2}\big) \leq CA^p$. This concludes the proof. □

4. Speed of convergence for the spliting method

The aim of this section is to show how the abstract convergence results obtained in Section 3 yield the convergence of a splitting method and extends the corresponding results from [10]. The proof, which is very similar to that in [10] is briefly sketched for the reader's convenience.

Assumption (A) *For $r = 0, \cdots, d_1$, let L_r be defined by (2.3) and for $l \geq 1$ let S_l be defined by (2.4). Suppose that the Assumptions* **(A2)** *and* **(A3(m+3))** *are satisfied, and that for every $\omega \in \Omega$ and r, l, $F_r(t) = F_r(t, \cdot)$ is a weakly continuous H^{m+3}-valued function and $G_l(t) = G_l(t, \cdot)$ is a weakly continuous H^{m+4}-valued function. Suppose furthermore that the initial condition $Z_0 \in L^2(\Omega; H^{m+3})$ is $\mathcal{F}_0$-measurable, that F_r and G_l are predictable and that for some constant K one has*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \sum_{r=0}^{d_1} \|F_r(t,\cdot)\|^p_{m+3} + \sup_{t\in[0,T]} \sum_{l\geq 1} \|G(t,\cdot)\|^p_{m+4} + \|Z_0\|^p_{m+3} \Big) \leq K.$$

Let $V^0 = (V^0_t, t \in [0,T])$ be a predictable, continuous increasing process such that $V^0_0 = 0$ and that there exists a constant K such that $V^0_T + \sum_{l\geq 1} \langle M^l \rangle_T \leq K$. Finally suppose that the following stochastic parabolicity condition holds.
For every $(t,x) \in [0,T] \times \mathbb{R}^d$, every $\omega \in \Omega$ and every $\lambda \in \mathbb{R}^d$,

$$\sum_{j,k=1}^{d} \lambda_j \lambda_k \Big[2a^{j,k}_0(t,x) dV^0_t + \sum_{l\geq 1} \sigma^j_l(t,x) \sigma^k_l(t,x) d\langle M \rangle_t \Big] \geq 0$$

in the sense of measures on $[0,T]$.

Let Z be the process solution to the evolution equation:

$$dZ(t,x) = \big(L_0 Z(t,x) + F_0(t,x)\big)dV_t^0 + \sum_{r=1}^{d_1} \big(L_r Z(t,x) + F_r(t,x)\big)dt + \sum_{l\geq 1} \big(S_l Z(t,x) + G_l(t,x)\big)dM_t^l \tag{4.1}$$

with the initial condition $Z(0,\cdot) = Z_0$. Then Theorem 1 proves the existence and uniquness of the solution to (4.1), and that

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|Z(t)\|_{m+3}^p\Big) \leq C$$

for some constant C which depends only on d, d_1, K, m, p and T.

For every integer $n \geq 1$ let $\mathcal{T}_n = \{t_i := iT/n, i = 0,1,\cdots,n\}$ denote a grid on the interval $[0,T]$ with constant mesh $\delta = T/n$. For $n \geq 1$, let $Z^{(n)}$ denote the approximation of Z defined for $t \in \mathcal{T}_n$ using the following splitting method: $Z^{(n)}n(0) = 0$ and for $i = 0,1,\cdots,n-1$, let

$$Z^{(n)}(t_{i+1}) := P_\delta^{(d_1)} \cdots P_\delta^{(2)} P_\delta^{(1)} Q_{t_i,t_{i+1}} Z^{(n)}(t_i), \tag{4.2}$$

where for $r = 1,\cdots,d_1$ and $t \in [0,T]$, $P_t^{(r)}\psi$ denotes the solution ζ_r of the evolution equation

$$d\zeta_r(t,x) = \big(L_r\zeta_r(t,x) + F_r(t,x)\big)dt \quad \text{and } \zeta_r(0,x) = \psi(x),$$

and for $s \in [0,t] \leq T$, $Q_{s,t}\psi$ denotes the solution η of the evolution equation defined on $[s,T]$ by the "initial" condition $\eta(s,x) = \psi(x)$ and for $t \in [s,T]$ by:

$$d\eta(t,x) = \big(L_0\eta(t,x) + F_0(t,x)\big)dV_t^0 + \sum_{l\geq 1} \big(S_l\eta(t,x) + G_l(t,x)\big)dM_t^l.$$

The following theorem gives the speed of convergence of this approximation.

Theorem 5. *Let $a_r^{j,k}, b_r^{j,k}, a_r^j, a_r, b_r, \sigma_l^j, \sigma_l, \tau_l, F_r, G_l$ satisfy the Assumption* **(A)**. *Suppose that $a_r^{j,k}, b_r^{j,k}, a_r^j, a_r, b_r$ and F_r are time-independent. Then there exists a constant $C > 0$ such that*

$$\mathbb{E}\Big(\sum_{t\in\mathcal{T}_n} \|Z^{(n)}(t) - Z(t)\|_m^p\Big) \leq C n^{-p}, \quad \textit{for every } n \geq 1.$$

Proof. Let $d' = d_1 + 1$ and let us introduce the following time change:

$$\kappa(t) = \begin{cases} 0 & \text{for } t \leq 0, \\ t - k\delta d_1 & \text{for } t \in [kd'\delta, (kd'+1)\delta),\ k = 0,1,\cdots,n-1, \\ (k+1)\delta & \text{for } t \in [(kd'+1)\delta, (k+1)d'\delta),\ k = 0,1,\cdots,n-1. \end{cases}$$

Let, for evry $t \in [0,T]$,

$$\begin{array}{lll} \tilde{M}^l(t) = M^l_{\kappa(t)}, & \tilde{\mathcal{F}}_t = \mathcal{F}_{\kappa(t)}, & \tilde{V}^0_{t,0} = \tilde{V}^0_{t,1} = V^0_{\kappa(t)}, \\ \tilde{V}^r_{t,0} = \kappa(t), & \tilde{V}^r_{t,1} = \kappa(t - r\delta) & \text{for } r = 1,2,\cdots,d_1. \end{array}$$

For $\varepsilon = 0, 1$, consider the evolution equations with the same initial condition $Z_0(0,x) = Z_1(0,x) = Z_0(x)$ and

$$dZ_\varepsilon(t) = \sum_{r=0}^{d_1} \big(L_r Z_\varepsilon(t) + F_r\big) d\tilde{V}^r_{t,\varepsilon} + \big(S_l Z_\varepsilon(t) + G_l\big) d\tilde{M}^l_t. \tag{4.3}$$

One easily checks that the Assumptions **(A1), (A2)**, **(A3(m+3)), (A4(m+3,p))** are satisfied with the martingales $\tilde{M}_l$ and the increasing processes $\tilde{V}^r_{\varepsilon,t}$ for $\varepsilon = 0, 1$ and $r = 0, 1, \cdots, d_1$. Therefore, Theorem 1 implies that for $\varepsilon = 0, 1$, the equation (4.3) has a unique solution. Furthermore, since condition (3.2) holds, Theorem 3 proves the existence of a constant C such that

$$\mathbb{E}\Big(\sup_{t\in[0,d'T]} \|Z_1(t) - Z_0(t)\|^p_m \Big) \le C \sup_{t\in[0,d'T]} \max_{1\le r\le d_1} |\kappa(t+r\delta) - \kappa(t)|^p = CT^p n^{-p}.$$

Since by construction, we have $Z_0(d't) = Z(t)$ and $Z_1(d't) = Z^{(n)}(t)$ for $t \in \mathcal{T}_n$, this concludes the proof. □

Note that the above theorem yields a splitting method for the following linear Schrödinger equation on $\mathbb{R}^d$:

$$\begin{aligned} dZ(t,x) =& \Big(i\Delta Z(t,x) + \sum_{j=1}^d a^j(x) D_j Z(t,x) + F(x) \Big) dt \\ &+ \sum_{l\ge 1} \big[\big(\sigma_l(x) + i\tau_l(x)\big) Z(t,x) + G_l(x) \big] dM^l_t, \end{aligned}$$

where a^j, F (resp. σ_l, τ_l and G_l) belong to H^{m+3} (resp. H^{m+4}). Indeed, this model is obtained with $a^{j,k} = \sigma^j_l = 0$ and $b^{j,k} = 1$ for $j, k = 1, \cdots, d$ and $l \ge 1$.

Finally, Theorem 4 yields the following theorem for the splitting method in the case of time demendent coefficients. The proof, similar to that of Theorem 5, will be omitted; see also [10], Theorem 5.2.

Theorem 6. *Let $a^{j,k}_r, b^{j,k}_r, a^j_r, \sigma^j_l, \sigma_l, \tau_l, F, G_l$ satisfy Assumptions* **(A)** *and* **(A5(m))**. *For every integer $n \ge 1$ let $Z^{(n)}$ be defined by (4.2) when the operators L_r, S_l, the processes F_r and G_l depend on time in a predictable way. Then there exists a constant $C > 0$ such that for every $n \ge 1$, we have:*

$$\mathbb{E}\Big(\sum_{t\in\mathcal{T}_n} \|Z^{(n)}(t) - Z(t)\|^p_m \Big) \le C n^{-p}.$$

References

[1] Blanes, S., Moan, P. C., 2000, Splitting methods for the time-dependent Schrödinger equation. *Phys. Lett. A* **265**, no. 1–2, 3542.

[2] Cardon-Weber, C., 2000. Implicit approximation scheme for the Cahn-Hilliard stochastic equation, Prépublication 613 du Laboratoire de Probabilités et Modèles Aléatoires.

[3] de Bouard, A., Debussche, A., 2004, A semi-discrete scheme for the nonlinear Schrödinger equation, *Num. Math.*, **96**, 733–770.

[4] de Bouard, A., Debussche, A., 2006, Weak and Stong order of Convergence for a Semidiscrete scheme for the Stochastic Nonlinear Schrödinger Equation, *Appl. Math. Optim.*, **54**, 369–399.

[5] Debussche, A., Printems, J., 1999. Numerical simulations for the stochastic Korteweg-de-Vries equation, *Physica D.* **134**, 200–226.

[6] Debussche, A., Printems, J., 2009, Weak order for the discretization of the stochastic heat equation. *Math. Comp.* **78**, no. 266, 845–863.

[7] Dujardin, G., Faou, E., 2008, Qualitative behavior of splitting methods for the linear Schrödinger equation in molecular dynamics. CANUM 2006—CongrÃ¨s National d'Analyse Numérique, 234–239, ESAIM Proc., **22**, EDP Sci., Les Ulis.

[8] Gyöngy, I., 1982 On stochastic equations with respect to semimartingales III, *Stochastics*, **7**, 231–254.

[9] Gyöngy, I., 1998. Lattice Approximations for Stochastic Quasi-Linear Parabolic Partial Differential Equations Driven by Space-Time White noise I, *Potential Analysis* **9**, 1–25 ; 1999, Lattice Approximations for Stochastic Quasi-Linear Parabolic Partial Differential Equations Driven by Space-Time White noise II, *Potential Analysis* **11**, 1–37.

[10] Gyöngy, I., Krylov, N.V., 2003 On the splitting-up method and stochastic partial differential equations *The Annals od Probability*, **31-2**, 564–591.

[11] Gyöngy, I., Krylov, N.V., 2005, An accelerated splitting-up method for parabolic equations. *SIAM J. Math. Anal.* **37** , no. 4, 1070–1097.

[12] Gyöngy, I., Millet, A., 2005, On Discretization Schemes for Stochastic Evolution Equations, *Potential Analysis*, **23**, 99–134.

[13] Gyöngy, I., Millet, A., 2009, Rate of convergence of space time approximations for stochastic evolution equations. Potential Anal. **30** , no. 1, 29–64.

[14] Hausenblas, E., 2003, Approximation for Semilinear Stochastic Evolution Equations, *Potential Analysis* **18**, no. 2, 141–186.

[15] Krylov, N.V. Rosovskii, B.L., 1981, Stochastic evolution equations, *J. Soviet Mathematics*, **16**, 1233–1277.

[16] Krylov, N.V., Rosovskii, B., On characteristics of degenerate second order parabolic Itô equations, *J. Soviet Mathh.*, **32**, 336–348.

[17] J.L. Lions and E. Magenes, Non-homogeneous boundary value problems and applications. Vol. I, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 181. Springer-Verlag, New York-Heidelberg, 1972.

[18] Millet, A., Morien, P.-L., 2005, On implicit and explicit discretization schemes for parabolic SPDEs in any dimension. Stochastic Process. Appl. 115, no. 7, 1073–1106.

[19] Neuhauser, C. Thalhammer, M., 2009, On the convergence of splitting methods for linear evolutionary Schrödinger equations involving an unbounded potential. BIT **49**, no. 1, 199–215.

[20] Pardoux, E., 1975, Équations aux dérivées partielles stochastiques nonlinéares monotones. Étude de solutions fortes de type Itô, Thèse Doct. Sci. Math. Univ. Paris Sud.

[21] Printems, J., 2001. On the discretization in time of parabolic stochastic differential equations, *Math. Model and Numer. Anal.* **35(6)**, 1055–1078.

Department of Mathematics, University of York, Heslington, York YO10 5DD, UK

E-mail address: zb500@york.ac.uk

Laboratoire de Probabilités et Modèles Aléatoires (CNRS UMR 7599), Universités Paris 6-Paris 7, Boîte Courrier 188, 4 place Jussieu, 75252 Paris Cedex 05, *and*, Centre d'Economie de la Sorbonne (CNRS UMR 8174), Équipe SAMOS-MATISSE, Université Paris 1 Panthéon Sorbonne, 90 Rue de Tolbiac, 75634 Paris Cedex 13

E-mail address: annie.millet@upmc.fr *and* annie.millet@univ-paris1.fr